MATLAB for Optimization

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Outline

- Introduction of mathematical optimization
- How to use optimization to model and solve engineering problems
- How to solve optimization models using MATLAB and CVX
- Applications
  - Production plan
  - Smart electric vehicle charging
  - Vehicle-to-grid
  - Cast some machine learning to optimization
  - Stochastic optimization
- Solving optimization problems

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  - Stochastic optimization
- Solving optimization problems
- The slides and all the codes are available at my website: http://uregina.ca/∼wang233z

What is an optimization model?

- A mathematical approach for seeking a “best” “decision/action” from a “set of alternatives”
- An objective function that is to be maximized or minimized
- A set of constraints (possibly empty) that must be satisfied

Formal Formulation

- Mathematical programs are problems of the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0,\ \forall i = 1, ..., m \end{aligned} \tag{1}$$

  - $x \in \mathbb{R}^n$ is the optimization variable
  - $f : \mathbb{R}^n \to \mathbb{R}$ is objective function
  - $g_i : \mathbb{R}^n \to \mathbb{R}$ are (inequality) constraint functions
- Feasible region: $\mathcal{C} = x : g_i(x) \le 0,\ \forall i = 1, ..., m$
- $x^\star \in \mathbb{R}^n$ is an optimal solution if $x^\star \in \mathcal{C}$, and $f(x^\star) \le f(x),\ \forall x \in \mathcal{C}$

Minimization vs Maximization

- Without loss of generality, it is sufficient to consider a minimization objective since
  $\underset{x}{\text{maximize}}\ \{f(x) : x \in \mathcal{C}\} \equiv -\underset{x}{\text{minimize}}\ \{f(x) : x \in \mathcal{C}\}$
- Thus to develop the theory we will only consider minimization problems. When actually solving problems we can use the actual min or max objective

Program vs Optimization Problem

- A “program” or “mathematical program” is an optimization problem with a finite number of variables and constraints written out using explicit mathematical (algebraic) expressions
- The word “program”, “programming” means “plan”, “planning”
- Early applications of optimization arose in planning resource allocations and gave rise to “programming” to mean optimization (predates computer programming)
- We will use “program”, “programming” and “optimization problem”, “optimization” interchangeably

Problem Classification

- The tractability of a large scale optimization problem depends on the structure of the functions that make up the objective and constraints, and the domain restrictions on the variables.

Functions	Variables	Problem type	Difficulty

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All linear	Continuous variables	Linear Program (LP) or Linear Optimization problem	easy

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Linear/nonlinear	Some discrete	Integer Program (IP) or Discrete optimization problem	Difficult

Develop optimization models

- Many problems of real importance can be formulated as an optimization problem
- There are two important aspects
  - Mathematical modeling of engineering problems
  - Problem solving
- Reducing a seemingly new problem to an instance of a well-known problem allows one to use pre-existing methods for solving them

Formulation Steps

- Encode decisions/actions as *decision variables* whose values we are seeking
- Identify the relevant *problem data*
- Express *constraints* on the values of the decision variables as mathematical relationships (inequalities) between the variables and problem data
- Express the *objective function* as a function of the decision variables and the problem data

Ex1: Production plan

- A company produces two types of drinks: Drink 1 and Drink 2 and they yield different profit.
- The capacity of the barrels to contain Drink 1 and Drink 2 are 500 L and 400 L.
- Each type drink requires some amount of ingredient A and B.
- There are 30 L ingredient A and 44 L ingredient B in stock.
- The manager is developing a plan to maximize the profit.

Drink (L)	**A (L)**	**B (L)**	**Profit ($)**
Drink 1	0.03	0.08	1.00
Drink 2	0.06	0.04	1.25

Formulation

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Formulation

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  i.e., x_1: Drink 1; x_2: Drink 2

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  - Ingredient constraints:

$$0.03x_1 + 0.08x_2 \leq 30$$

$$0.08x_1 + 0.04x_2 \leq 44$$

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- *Data*: required ingredient to produce drinks, amount of ingredient in stock, barrel capacity, profit of each drink
- *Constraints*:
  - Ingredient constraints:
    $$0.03x_1 + 0.08x_2 \leq 30$$
    $$0.08x_1 + 0.04x_2 \leq 44$$
  - Barrel constraints: $x_1 \leq 500, x_2 \leq 400$
  - Production constraints: $x_1 \geq 0, x_2 \geq 0$
- *Objective function*: $1x_1 + 1.25x_2$

Formulation

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & -(1x_1 + 1.25x_2) \\
\text{subject to} \quad & 0.03x_1 + 0.06x_2 \leq 30 \\
& 0.08x_1 + 0.04x_2 \leq 44 \\
& x_1 \leq 500 \\
& x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned}
\tag{2}
$$

Formulation

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & -(1x_1 + 1.25x_2) \\
\text{subject to} \quad & 0.03x_1 + 0.06x_2 \leq 30 \\
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- Linear programming

Formulation

$$
\begin{aligned}
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\text{subject to} \quad & 0.03x_1 + 0.06x_2 \leq 30 \\
& 0.08x_1 + 0.04x_2 \leq 44 \\
& x_1 \leq 500 \\
& x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned} \tag{2}
$$

- Linear programming
- can be solve by $linprog$ using MATLAB

MATLAB's Optimization Toolbox

- Not a complete list

Problem type	**Solvers**
Linear Programming	$linprog$
Mixed Linear Programming	$intlinprog$
Linear Least Squares	$lsqlin, lsqnonneg$
Quadratic Programming	$quadprog$

Linear programming solver

- $linprog(c, A, b, Aeq, beq, lb, ub, options)$
- Finds the minimum of a problem specified by

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & A_{eq}x = b_{eq} \\ & lb \leq x \leq ub \end{aligned} \tag{3}$$

where c, x, b, beq, lb, and ub are vectors, and A and Aeq are matrices.

Model reformulation

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & 1x_1 + 1.25x_2 \\
\text{subject to} \quad & 0.03x_1 + 0.06x_2 \leq 30 \\
& 0.08x_1 + 0.04x_2 \leq 44 \\
& x_1 \leq 500 \\
& x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

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$$

Model reformulation

- What are A, b, c, x?

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$$A = \begin{bmatrix} 0.03 & 0.06 \\ 0.08 & 0.04 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 30 \\ 44 \\ 500 \\ 400 \\ 0 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 1.25 \end{bmatrix}$$

Model reformulation

- What are A, b, c, x?

$$A = \begin{bmatrix} 0.03 & 0.06 \\ 0.08 & 0.04 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 30 \\ 44 \\ 500 \\ 400 \\ 0 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 1.25 \end{bmatrix}$$

- Matlab code: $x = linprog(c, A, b)$
  here:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Model reformulation (alternative)

- $x = linprog(c, A, b, [\,], [\,], lb, ub)$

$$A = \begin{bmatrix} 0.03 & 0.06 \\ 0.08 & 0.04 \end{bmatrix} \; b = \begin{bmatrix} 30 \\ 44 \end{bmatrix} \; c = \begin{bmatrix} 1 \\ 1.25 \end{bmatrix}$$

$$lb = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; ub = \begin{bmatrix} 500 \\ 400 \end{bmatrix} \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Quadratic programming solver

- $quadprog(H, c, A, b, Aeq, beq, lb, ub)$
- Finds the minimum of a problem specified by

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & \frac{1}{2}x^T Hx + c^T x \\
\text{subject to} \quad & Ax \leq b \\
& A_{eq}x = b_{eq} \\
& lb \leq x \leq ub
\end{aligned}
\tag{4}
$$

  where c, x, b, beq, lb, and ub are vectors, and H, A and Aeq are matrices.
- It is harder to reformulate a problem to standard quadratic program.

CVX

- CVX is a Matlab-based modeling system for convex optimization.
- CVX turns Matlab into a modeling language, allowing constraints and objectives to be specified using standard Matlab expression syntax.

CVX code

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & 1x_1 + 1.25x_2 \\
\text{subject to} \quad & 0.03x_1 + 0.06x_2 \leq 30 \\
& 0.08x_1 + 0.04x_2 \leq 44 \\
& x_1 \leq 500 \\
& x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned}
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CVX code

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\begin{aligned}
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\end{aligned}
$$

```
cvx_begin
  variable x(2);
  minimize -(1*x(1)+
    1.25*x(2))
  subject to
  0.03*x(1)+0.06*x(2)<=30;
  0.08*x(1)+0.04*x(2)<=44;
  x(1)<=500;
  x(2)<=400;
  x(1)>=0;
  x(2)>=0;
cvx_end
```

CVX Python code

```
import numpy as np
import cvxpy as cp
n = 2
x = cp.Variable(n)
objective = cp.Minimize(-(100*x[0]+125*x[1]))
constraints = [3*x[0]+6*x[1]<=30,
               8*x[0]+4*x[1]<=44,
               x[0]<=5,
               x[1]<=4,
               x[0]>=0,
               x[1]>=0]
prob = cp.Problem(objective, constraints)
result = prob.solve()
print(x.value)
```

Electric vehicles (EV)

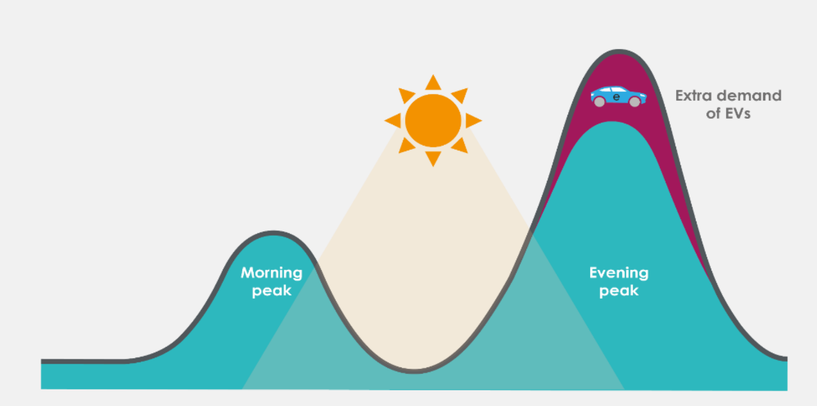
- Lower maintenance costs
- Lower taxes
- Cheaper fuel (electricity)
- Government subsidy

Ref: fleetcarma.com

Basic EV charging

- Typical battery capacity: $30kWh - 100kWh$
- Onboard charger: 1.9 – 22 kW
- DC offboard charger: 50 – 350 kW
- Charging level
  - Level 1: 0 – 10 kW
  - Level 2: 10 – 50 kW
  - Level 3: 50 – 350 kW

Impact on power system



Ref: Electric Cars, edx.org

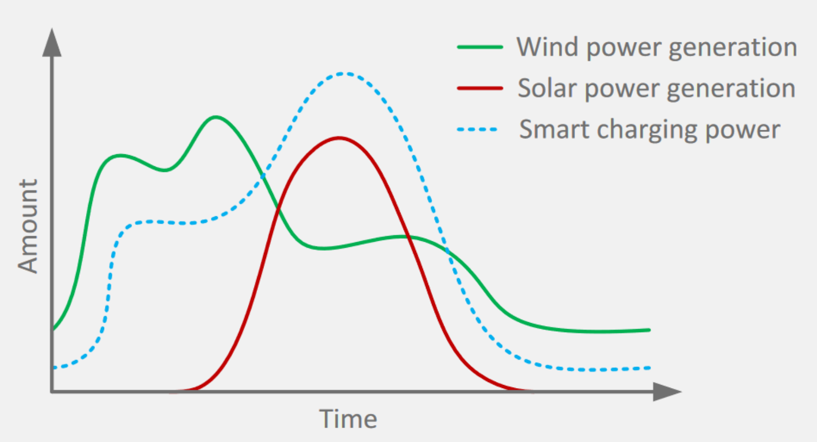
Smart charging and vehicle-to-grid (V2G)

- Using the electric vehicle battery to feed power back to the grid using a bidirectional EV charger
- Advantages: storage for renewables, reduce peak demand, ancillary services, etc
- Challenges: optimal control, bidirectional charger, battery degradation, standardization, regulatory framework, etc

V2G applications

- Local load balancing
  - Adjust charging time/power according to load
  - Balance multiple charge points with priority
- Renewable energy utilization
- Price based charging/discharging
- Peak shaving
- Grid back up

Renewable energy availability



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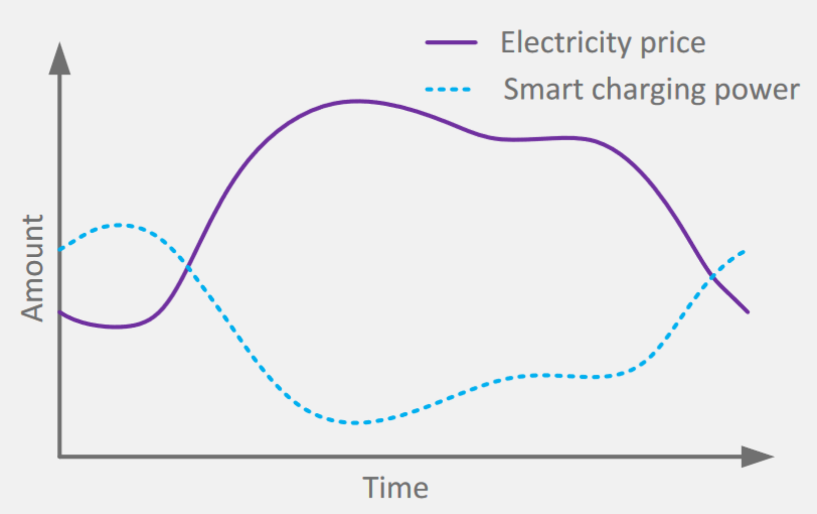
Price based charging

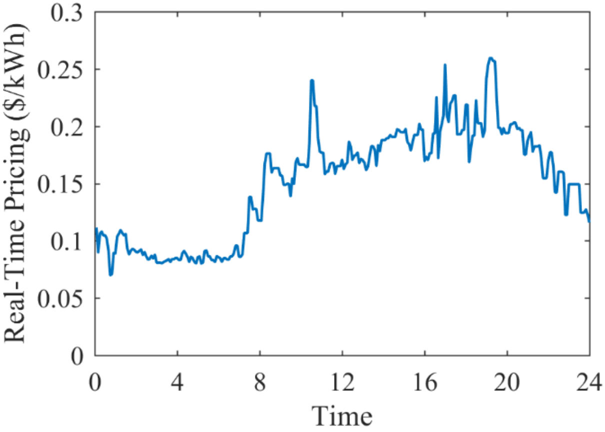


Ref: Electric Cars, edx.org

Ex2: Smart charging

- You have an EV and can be charged at home.
- The electricity price is real-time pricing (RTP).
- You need $20kW$ to charge up your EV.
- The rated power of the charger is $2kW$.
- Assume that the EV can be charged anytime in the day.
- Now, you need to determine what time to charge to minimize the electricity bill in the day.

Real-time-pricing



Ref: Calculated from locational marginal price on October 27, 2019 in PJM market

Smart charging model

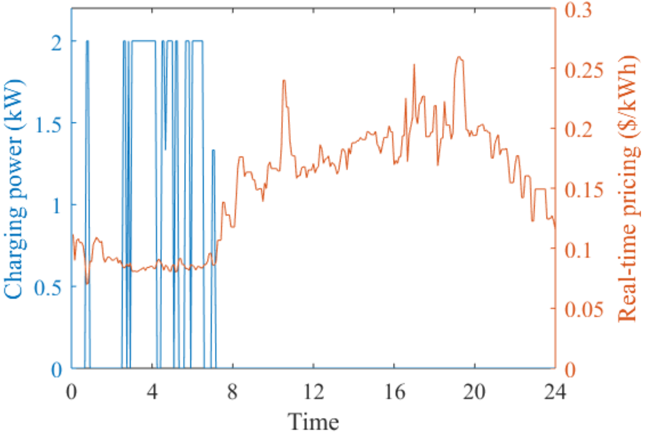
$$\begin{aligned} \underset{t}{\text{minimize}} \quad & \sum_{t \in \mathcal{T}} \pi_t p_t \\ \text{subject to} \quad & \sum_{t \in \mathcal{T}} p_t = 20 \\ & 0 \leq p_t \leq 2 \end{aligned}$$

where π_t is price, p_t is charging power.

CVX code

```
load('As3_data')
price = price';
cvx_begin
    variable p(288)
    minimize price * p
    subject to
        sum(p)==20*60/15; %kW15min
        0<=p<=2;
cvx_end
```

Smart charging



Smart charging?

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Smart charging?

- Is this too aggressive?
- How about battery degradation cost?
- How about
  - home arriving, leaving and driving time?
  - state of charge?
  - multiple vehicles?
  - convenience and privacy?
  - power system reliability?

Ex3: V2G

$$\underset{P_t^{EV_i}}{\text{minimize}}\ \operatorname{var}\left(P_t^{EV_i} - P_{t\in[t_1,t_2]\cup[t_3,t_4]}^{EV_i} + P_t^{EV_{-i}} + P_{base,t}\right)$$

$$+\,\upsilon \sum_{t\in\mathcal{T}} DC(DOD_t)\left|P_t^{EV_i}\right| + \kappa \sum_{t\in\mathcal{T}} RTP_t\ P_t^{EV_i}$$

subject to:

$$-P_{rated} \leq P_t^{EV_i} \leq P_{rated}, \forall\, t \in [t_2, t_3] \cup [t_4, t_1]$$

$$\sum_t P_t^{EV_i} = \beta E_0, \ \forall\, t \in [t_1, t_2] \cup [t_3, t_4]$$

$$SOC_t = SOC_0 + \sum_{t=0}^{t} P_t^{EV_i}$$

$$SOC_{min} \leq SOC_t \leq SOC_{max}, \forall\, t \in \mathcal{T}$$

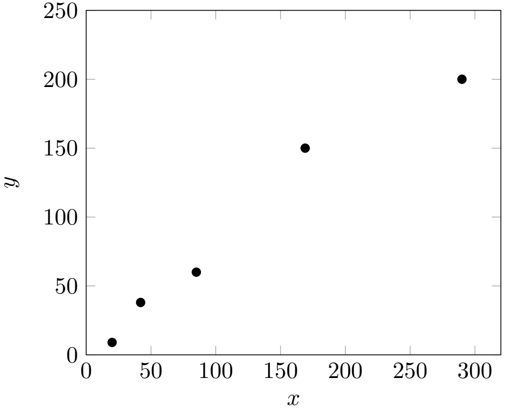
$$SOC_{t_2} \geq SOC_{acc}$$

$$SOC_{t_4} \geq SOC_{acc}$$

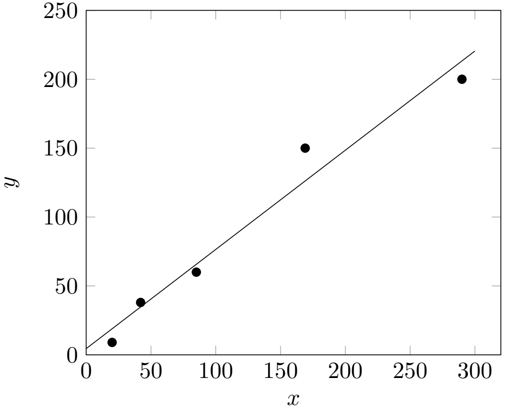
$$DOD_t = 1 - SOC_t, \quad \forall\, t \in \mathcal{T}$$

Ref: K. Ginigeme and Z. Wang, "Distributed Optimal Vehicle-To-Grid Approaches with Consideration of Battery Degradation Cost under Real-Time Pricing", IEEE Access, 2020.

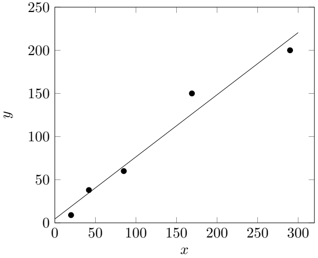
Machine learning: data fitting



Machine learning: data fitting



Machine learning: data fitting



- Given $x_i, y_i, i = 1, ..., m$, find $f(\theta) = \theta_1 * x + \theta_2$ that optimizes

$$\underset{\theta}{\text{minimize}} \sum_{i=1}^{m} (\theta_1 x_i + \theta_2 - y_i)^2$$

where θ_1 is slope, θ_2 is intercept.

Formal problem setting

- Input: $x_i \in \mathbb{R}^n, i = 1, ..., m$
- Output: $y_i \in \mathbb{R}, i = 1, ..., m$
- Model parameters: $\theta \in \mathbb{R}^k$
- Predicted output: $\hat{y} \in \mathbb{R}$

Formal problem setting

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- Model parameters: $\theta \in \mathbb{R}^k$
- Predicted output: $\hat{y} \in \mathbb{R}$
- Let's define a function that maps inputs to feature vectors

$$\phi : \mathbb{R}^n \to \mathbb{R}^k$$

- Then, we can write

$$\hat{y}_i = \sum_{j=1}^{k} \theta_j \cdot \phi_j(x_i) \equiv \theta^T \phi(x_i)$$

Formal problem setting

$$\hat{y}_i = \sum_{i=1}^{j} \theta_j \cdot \phi_j(x_i) \equiv \theta^T \phi(x_i)$$

- Let's write it in a more compact way

$$\Phi \in \mathbb{R}^{m \times k} = \begin{bmatrix} \phi(x_1)^T \\ \phi(x_2)^T \\ \vdots \\ \phi(x_m)^T \end{bmatrix}, \; y \in \mathbb{R}^m = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Then,

$$\hat{y} = \Phi\theta$$

Loss function

- The loss function is defined as

$$\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$$

- Square loss

$$\ell(\hat{y}, y) = (\hat{y} - y)^2$$

- Absolute loss

$$\ell(\hat{y}, y) = |\hat{y} - y|$$

- Deadband loss

$$\ell(\hat{y}, y) = max\{0, |\hat{y} - y| - \varepsilon\}, \varepsilon \in \mathbb{R}_+$$

Data fitting with square Loss

- Square loss

$$\ell(\hat{y}, y) = (\hat{y} - y)^2$$

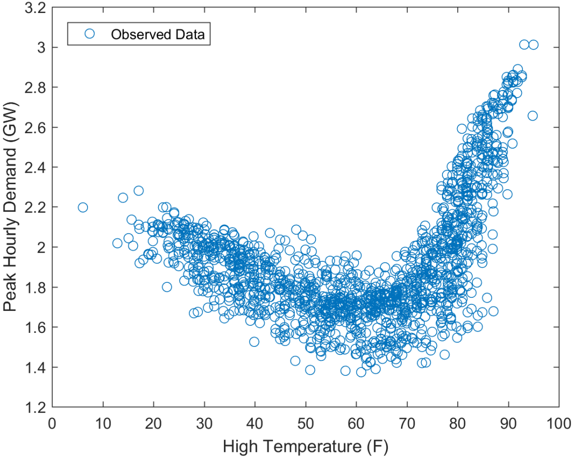
- Called least-squares objective function

$$\underset{\theta}{\text{minimize}}\ (\hat{y} - y)^2 = \underset{\theta}{\text{minimize}}\ \|\hat{y} - y\|_2^2$$

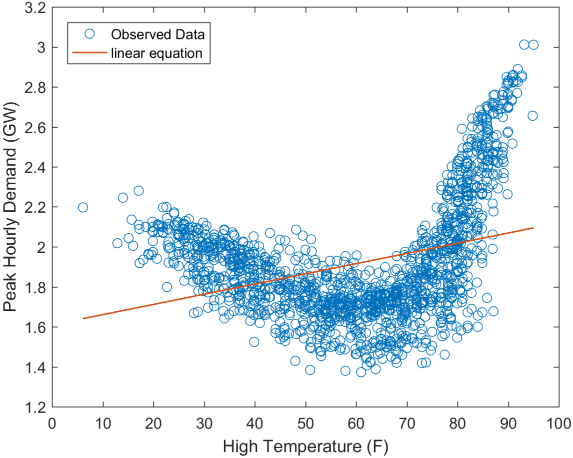
- Has analytical solution

$$\theta^\star = (\Phi^T \Phi)^{-1} \Phi^T y$$

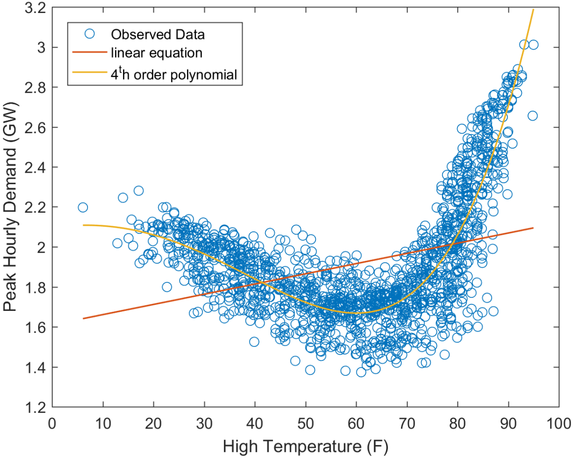
Ex4: Electricity peak demand forecasting



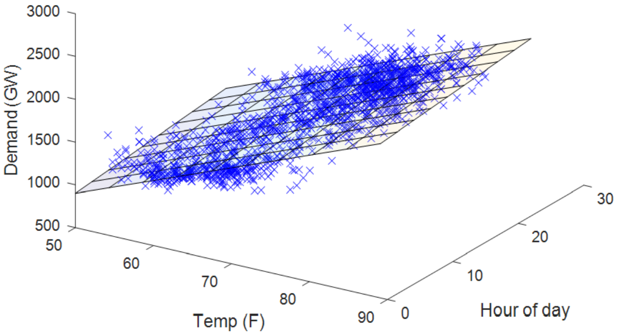
Ex4: Electricity peak demand forecasting



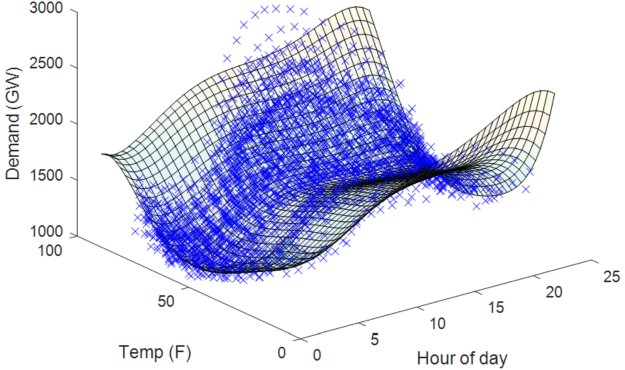
Ex4: Electricity peak demand forecasting



Linear regression 2d vector



"Nonlinear regression" 2d vector



Matlab code

```
X = load('max_temp_b.txt');
y = load('max_demand_b.txt');
[m,n] = size(X);

%% linear
Phi = [X ones(m,1)];
% theta = inv(Phi' * Phi) * Phi' * y;
theta = Phi \ y;

%% 4th order polynomial
Phi = [X.^4 X.^3 X.^2 X ones(m,1)];
theta = Phi \ y;
```

Ex5: Newsvendor problem

- A company produces winter coats.
- The company must commit to specific production quantity x before knowing the exact demand d, 3 months before the winter season.

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- After seeing demand d, the company decides the amount y_r to sell in regular price π_r, and the amount y_s to sell at a salvage/discounted price π_s.
- This is called decision making under *uncertainty*, because decision x is made under uncertain demand d.

Two stage stochastic programming

- Decision variables:
  - Here-and-Now decision: production quantity x
  - Wait-and-See decision: regular price quantity y_r, discounted price quantity y_s
- Objective: minimize production cost and expected future cost

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$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) + \mathbb{E}_d[Q(x,d)] \\ \text{subject to} \quad & 0 \leq x \leq \hat{x} \end{aligned}$$

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\end{aligned}
$$

$$
\begin{aligned}
Q(x, d) = \underset{y_r(d), y_s(d)}{\text{minimize}} \quad & -(\pi_r \; y_r(d) + \pi_s \; y_s(d)) \\
\text{subject to} \quad & y_r(d) \le d, \forall d \in \mathcal{D} \\
& y_r(d) + y_s(d) \le x, \forall d \in \mathcal{D} \\
& y_r(d), y_s(d) \ge 0, \forall d \in \mathcal{D}
\end{aligned}
$$

Reformulation

- Combine the two stages

$$
\begin{array}{ll}
\underset{x,\, y_r(d),\, y_s(d)}{\text{minimize}} & f(x) + \mathbb{E}_d[-(\pi_r\ y_r(d) + \pi_s\ y_s(d))] \\
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- Suppose demand d is a discrete random variable with S scenarios $(d_1, ..., d_s)$, and each scenario d_i with a probability p_i.
- Correspondingly, the sell quantities have $y_{r,i}$ and $y_{s,i}$ for each scenario d_i.

Reformulation

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& y_r(d), y_s(d) \ge 0, \forall d \in \mathcal{D}
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\underset{x,\, y_{r,i},\, y_{s,i}}{\text{minimize}} \quad & cx - \sum_{i=1}^{S} p_i(\pi_r\ y_{r,i} + \pi_s\ y_{s,i}) \\
\text{subject to} \quad & 0 \le x \le \hat{x} \\
& y_{r,i} \le d_i, \forall i = 1, ..., S \\
& y_{r,i} + y_{s,i} = x, \forall i = 1, ..., S \\
& y_{r,i}, y_{s,i} \ge 0, \forall i = 1, ..., S
\end{aligned}$$

A concrete example

- Suppose there are 3 scenarios, $d_1 = 10$ with probability of $\frac{1}{4}$; $d_2 = 30$ with probability of $\frac{5}{12}$; $d_3 = 50$ with probability of $\frac{1}{3}$.
- Unit cost to produce coats: $c = 5$, regular price: $\pi_r = 10$, discounted price: $\pi_s = 3$.
- Production capacity: $\hat{x} = 70$.

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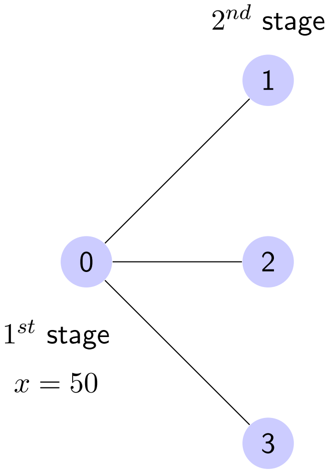
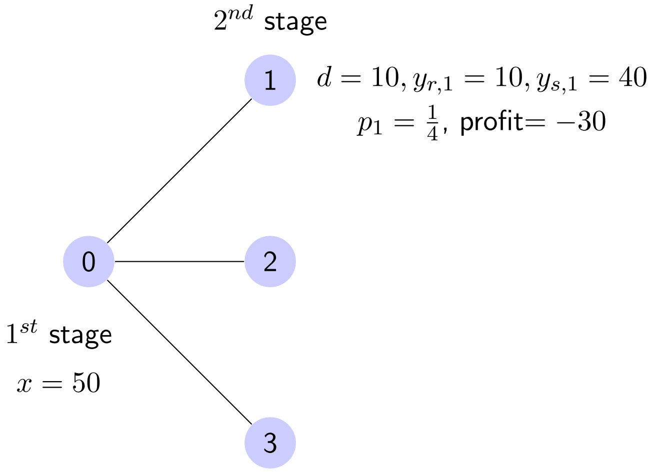
- Unit cost to produce coats: $c = 5$, regular price: $\pi_r = 10$, discounted price: $\pi_s = 3$.
- Production capacity: $\hat{x} = 70$.

$$\begin{aligned}
\underset{x,\, y_{r,i},\, y_{s,i}}{\text{minimize}} \quad & 5x - [\frac{1}{4}(10y_{r,1} + 3y_{s,1}) + \frac{5}{12}(10y_{r,2} + 3y_{s,2}) \\
& + \frac{1}{3}(10y_{r,3} + 3y_{s,3})] \\
\text{subject to} \quad & 0 \leq x \leq 70 \\
& y_{r,1} \leq 10, y_{r,2} \leq 30, y_{r,3} \leq 50 \\
& y_{r,i} + y_{s,i} \leq x, \forall i = 1, ..., S \\
& y_{r,i}, y_{s,i} \geq 0, \forall i = 1, ..., S
\end{aligned}$$

Results



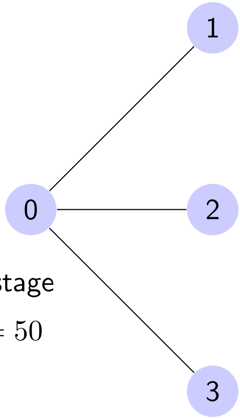
Results



2^{nd} stage
1
$d = 10, y_{r,1} = 10, y_{s,1} = 40$
$p_1 = \frac{1}{4}$, profit$= -30$
0
2
3
1^{st} stage
$x = 50$

Results

2^{nd} stage



$d = 10, y_{r,1} = 10, y_{s,1} = 40$

$p_1 = \frac{1}{4}$, profit$= -30$

$d = 30, y_{r,2} = 30, y_{s,2} = 20$

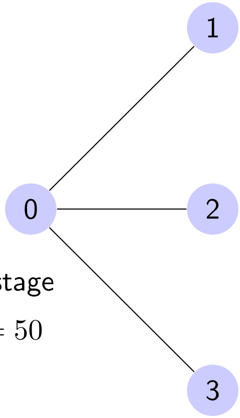
$p_2 = \frac{5}{12}$, profit$=110$

1^{st} stage

$x = 50$

Results

2^{nd} stage



$d = 10, y_{r,1} = 10, y_{s,1} = 40$

$p_1 = \frac{1}{4}$, profit$= -30$

$d = 30, y_{r,2} = 30, y_{s,2} = 20$

$p_2 = \frac{5}{12}$, profit$=110$

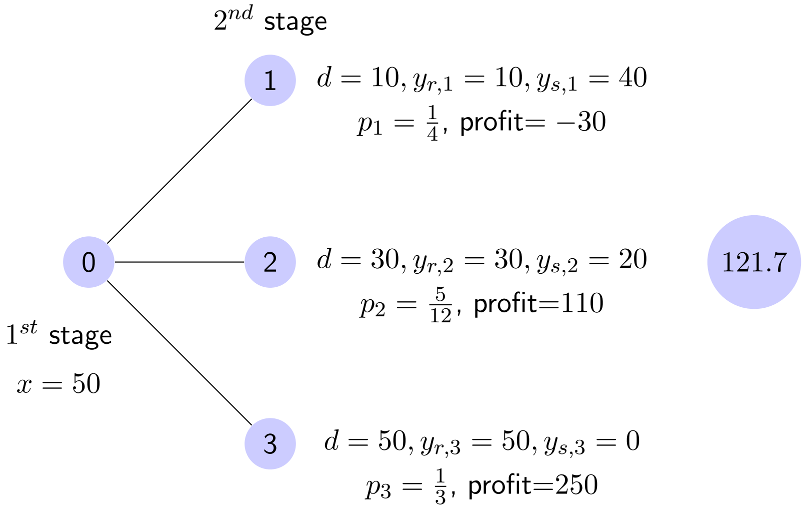
$d = 50, y_{r,3} = 50, y_{s,3} = 0$

$p_3 = \frac{1}{3}$, profit$=250$

1^{st} stage

$x = 50$

Results


2^{nd} stage
1
$d = 10, y_{r,1} = 10, y_{s,1} = 40$
$p_1 = \frac{1}{4}$, profit$= -30$
0
2
$d = 30, y_{r,2} = 30, y_{s,2} = 20$
$p_2 = \frac{5}{12}$, profit$=110$
121.7
1^{st} stage
$x = 50$
3
$d = 50, y_{r,3} = 50, y_{s,3} = 0$
$p_3 = \frac{1}{3}$, profit$=250$

Dealing with the General Case

- What if it is hard to define scenarios?

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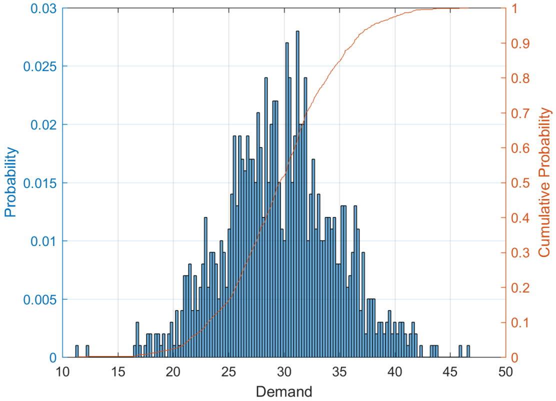
- What if our distribution is not discrete?

Dealing with the General Case

- What if it is hard to define scenarios?
- What if our distribution is not discrete?
- Sampling is enough under most general conditions



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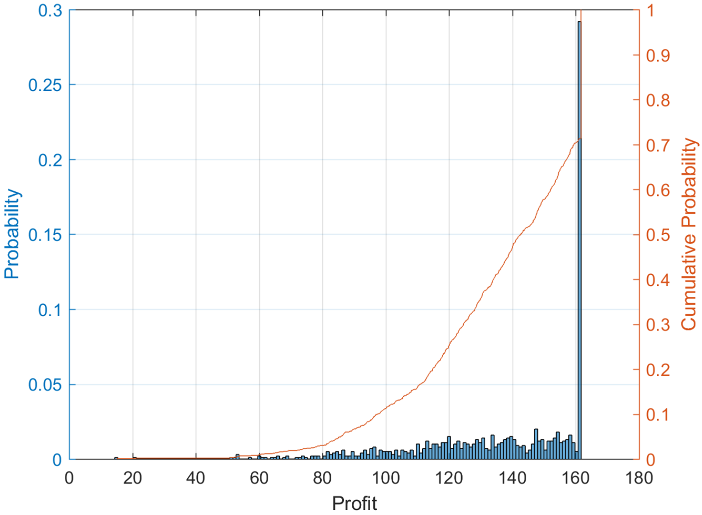

0.3
0.25
0.2
0.15
0.1
0.05
0
Probability
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Cumulative Probability
0
20
40
60
80
100
120
140
160
180
Profit

Electricity market-clearing problem

- A two-settlement market clearing problem with day ahead (DA) and real time (RT) stages

$$
\begin{aligned}
\underset{p_g^{DA}, p_{g,\omega}^{RT}, p_{d,\omega}^{shed},}{\text{minimize}} \quad & cost^{DA}(p_g^{DA}) + \mathbb{E}_\omega[cost^{RT}(p_{g,\omega}^{RT}, p_{d,\omega}^{shed})] \\
\text{subject to} \quad & f(p_g^{DA}) \leq 0 \\
& g(p_{g,\omega}^{RT}, p_{d,\omega}^{shed}) \leq 0, \ \forall \omega \\
& g(p_g^{DA}, p_{g,\omega}^{RT}, p_{d,\omega}^{shed}) \leq 0, \ \forall \omega
\end{aligned}
$$

How is the optimization problem actually solved?

- Unconstrained problem. Set the derivative (one dimensional) or gradient (high dimensional)

$$\nabla_x f(x) = 0$$

- Then how to find $\nabla_x f(x) = 0$
  - Direct solution: Analytically compute
  - Gradient descent

$$\text{Repeat: } x \leftarrow x - \alpha \nabla_x f(x)$$

  - Newton's method

$$\text{Repeat: } x \leftarrow x - \left(\nabla_x^2 f(x)\right)^{-1} \nabla_x f(x)$$

Constrained optimization

- Barrier method: Approximate problem via unconstrained optimization

$$\underset{x}{\text{minimize}}\ f(x) - t\sum_{i=1}^{m} log(-g_i(x))$$

as $t \to 0$, this approaches original problem

- Maximize Lagrangian dual problem

$$\underset{\lambda}{\text{maximize}}\ \{\underset{x}{\text{minimize}}\ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}$$

Practically solving optimization problems

- The good news, for many classes of optimization problems, people have already done all the "hard work" of developing numerical algorithms
- A wide range of tools that can take optimization problems in "natural" forms and compute a solution
- Some well-known libraries: CVX (MATLAB), CVXPY (Python), YALMIP(MATLAB), AMPL (custom language), GAMS (custom language), Gurobi (custom language)

Brief history of convex optimization

- Theory (convex analysis): ca1900–1970
- Algorithms
  - 1947: simplex algorithm for linear programming (Dantzig)
  - 1960s: early interior-point methods (Fiacco & McCormick, Dikin, . . .)
  - 1970s: ellipsoid method and other subgradient methods
  - 1980s: polynomial-time interior-point methods for linear programming (Karmarkar 1984)
  - late 1980s–now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski 1994)
- Applications
  - before 1990: mostly in operations research; few in engineering
  - since 1990: many new applications in engineering (control, signal
  - processing, communications, circuit design, . . .); new problem classes (semidefinite and second-order cone programming, robust optimization)

MATLAB history

- Invented by Prof. Cleve Moler, University of New Mexico in late 1970s
- The MathWorks, Inc. was formed in 1984 by Moler and Jack little. One product: MATLAB
- Today: 100 products; over 1 million users worldwide.

Take home messages

- Many problems of real importance can be formulated as an optimization problem.
- Particularly, convex optimization problems that can be solved efficiently and which still find a huge number of applications
- Reducing a seemingly new problem to an instance of a well-known problem allows one to use pre-existing methods for solving them

References

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Thank you!