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ORIGINAL PAPER

# On complete convergence for widely orthant-dependent random variables and its applications in nonparametric regression models

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**Abstract** In this paper, some probability inequalities and moment inequalities for widely orthant-dependent (WOD, in short) random variables are presented, especially the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality. By using these inequalities, we further study the complete convergence for weighted sums of arrays of row-wise WOD random variables and give some special cases, which extend some corresponding ones for dependent sequences. As applications, we present some sufficient conditions to prove the complete consistency for the estimator of nonparametric regression model based on WOD errors by using the complete convergence that we established. At last, the choice of the fixed design points and the weight functions for the nearest neighbor estimates is proposed. Our results generalize some known results for independent random variables and some dependent random variables.

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## 1 Introduction

It is well known that the probability limit theorem and its applications for independent random variables have been studied by many authors, while the assumption of independence is not reasonable in real practice. If the independent case is classical in the literature, the treatment of dependent random variables is more recent.

One of the important dependence structure is the wide dependence structure, which was introduced by Wang et al. (2013) as follows.

**Definition 1.1** For the random variables $\{X_n, n \geq 1\}$, if there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} P(X_i > x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely upper orthant dependent (WUOD, in short); if there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^{n} P(X_i \leq x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the $\{X_n, n \geq 1\}$ are widely orthant dependent (WOD, in short), and $g_U(n)$, $g_L(n)$, $n \geq 1$, are called dominating coefficients.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is called row-wise WOD if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of WOD random variables.

Recall that when $g_L(n) = g_U(n) = M$ for some constant $M$, the random variables $\{X_n, n \geq 1\}$ are called extended negatively upper orthant dependent (ENUOD, in short) and extended negatively lower orthant dependent (ENLOD, in short), respectively. If they are both ENUOD and ENLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are extended negatively orthant dependent (ENOD, in short). The concept of general extended negative dependence was proposed by Liu (2009), Liu (2010) and further promoted by Chen et al. (2010), Chen et al. (2011), Shen (2011), Shen (2013a), Wang and Cheng (2011), Wang and Wang (2012), and so forth. When $g_L(n) = g_U(n) = 1$ for any $n \geq 1$, the random variables $\{X_n, n \geq 1\}$ are called negatively upper orthant dependent (NUOD, in short) and negatively lower orthant dependent (NLOD, in short), respectively. If they are both NUOD and NLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are negatively orthant dependent (NOD, in short). The concept of negative dependence was introduced by Ebrahimi and Ghosh (1981) and carefully studied by Joag and Proschan (1983). For more details about NOD random variables, one can refer to Wang et al. (2010, 2011a,b), Wu (2006, 2010), Wu and Jiang (2011), Sung (2011), Qiu et al. (2011), and so forth. Joag and Proschan (1983) pointed out that NA random variables are NOD. Hu (2000) introduced the concept of negatively superadditive dependence (NSD, in short) and gave an example illustrating that NSD does not imply NA. Hu (2000) posed an open problem whether NA implies NSD. Christofides and Vaggelatou (2004) solved this open problem and indicated that NA implies NSD. In addition, Hu (2000) pointed out that NSD implies NOD (see Property 2 of Hu 2000). By the statements above, we can see that the class of WOD random variables contains END random variables, NOD random variables, NSD random variables, NA random variables and independent random variables as special cases. Hence, studying the probability limiting behavior of WOD random variables and its applications are of great interest.

The concept of WOD random variables was introduced by Wang et al. (2013) and many applications have been found subsequently. See, for example, Wang et al. (2013) provided some examples which showed that the class of WOD random variables contains some common negatively dependent random variables, some positively dependent random variables and some others; in addition, they studied the uniform asymptotics for the finite-time ruin probability of a new dependent risk model with a constant interest rate. Wang and Cheng (2011) presented some basic renewal theorems for a random walk with widely dependent increments and gave some applications. Wang et al. (2012) studied the asymptotics of the finite-time ruin probability for a generalized renewal risk model with independent strong subexponential claim sizes and widely lower orthant dependent inter-occurrence times. Liu (2012) gave the asymptotically equivalent formula for the finite-time ruin probability under a dependent risk model with constant interest rate. He et al. (2013) provided the asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables. Chen et al. (2013) considered uniform asymptotics for the finite-time ruin probabilities of two kinds of nonstandard bidimensional renewal risk models with constant interest forces and diffusion generated by Brownian motions. Shen (2013b) established the Bernstein type inequality for WOD random variables and gave some applications, and so forth.

The main purpose of the paper is to present some probability inequalities and moment inequalities for WOD random variables, especially the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality. By using these probability inequalities and moment inequalities, we further study the complete convergence for arrays of row-wise WOD random variables. In addition, we will apply the complete convergence to nonparametric regression model and investigate the complete consistency for the nonparametric regression estimator based on WOD errors.

The following concept of stochastic domination will be used in this work.

**Definition 1.2** An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable $X$ if there exists a positive constant $C$ such that

$$P(|X_{ni}| > x) \leq CP(|X| > x)$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

Throughout the paper, let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with dominating coefficients $g_U(n)$, $g_L(n)$, $n \geq 1$. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of row-wise WOD random variables with dominating coefficients $g_U(n)$, $g_L(n), n \geq 1$ in each row, where $\{k_n, n \geq 1\}$ is a sequence of positive integers. Denote $g(n) = \max\{g_U(n), g_L(n)\}$, $S_n = \sum_{i=1}^n X_i$ and $M_{t,n} = \sum_{i=1}^n E|X_i|^t$ for some $t > 0$ and each $n \geq 1$. Let $C$ denote a positive constant, which can be different in various places. $\lceil x \rceil$ denotes the integer part of $x$.

The structure of the paper is as follows: some important probability inequalities and moment inequalities are presented in Sect. 2. The complete convergence for arrays of row-wise WOD random variables are studied in Sect. 3 and the complete consistency for the estimator of nonparametric regression model based on WOD errors is investigated in Sect. 4.

## 2 Inequalities for WOD random variables

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property for WOD random variables, which was obtained by Wang et al. (2013)

**Lemma 2.1** (i) *Let $\{X_n, n \geq 1\}$ be WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$). If $\{f_n(\cdot), n \geq 1\}$ are nondecreasing, then $\{f_n(X_n), n \geq 1\}$ are still WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$); if $\{f_n(\cdot), n \geq 1\}$ are nonincreasing, then $\{f_n(X_n), n \geq 1\}$ are WUOD (WLOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$).*

(ii) *If $\{X_n, n \geq 1\}$ are nonnegative and WUOD with dominating coefficients $g_U(n), n \geq 1$, then for each $n \geq 1$,*

$$E\prod_{i=1}^{n} X_i \leq g_U(n)\prod_{i=1}^{n} EX_i.$$

*In particular, if $\{X_n, n \geq 1\}$ are WUOD with dominating coefficients $g_U(n), n \geq 1$, then for each $n \geq 1$ and any $s > 0$,*

$$E\exp\left\{s\sum_{i=1}^{n} X_i\right\} \leq g_U(n)\prod_{i=1}^{n} E\exp\{sX_i\}.$$

By Lemma 2.1, we can get the following corollary immediately.

**Corollary 2.1** *Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables.*

(i) *If $\{f_n(\cdot), n \geq 1\}$ are all nondecreasing (or all nonincreasing), then $\{f_n(X_n), n \geq 1\}$ are still WOD.*

(ii) *For each* $n \geq 1$ *and any* $s \in \mathrm{R}$,

$$E \exp\left\{s \sum_{i=1}^{n} X_i\right\} \leq g(n) \prod_{i=1}^{n} E \exp\{s X_i\}.$$

In the following, we will present some probability inequalities and moment inequalities for WOD random variables. Inspired by Fakoor and Azarnoosh (2005), Asadian et al. (2006) and Shen (2011), we can get the following probability inequality for WOD random variables.

**Lemma 2.2** *Let* $0 < t \leq 2$ *and* $\{X_n, n \geq 1\}$ *be a sequence of WOD random variables. Assume further that* $EX_n = 0$ *for each* $n \geq 1$ *when* $1 \leq t \leq 2$. *Then for all* $x > 0$ *and* $y > 0$,

$$P(|S_n| \geq x) \leq \sum_{i=1}^{n} P(|X_i| \geq y) + 2g(n) \exp\left\{\frac{x}{y} - \frac{x}{y} \ln\left(1 + \frac{xy^{t-1}}{M_{t,n}}\right)\right\}. \quad (1)$$

*Proof* If $0 < t \leq 1$, then we can get (1) by the similar proof of Theorem 2.3 in Shen (2011). If $1 \leq t \leq 2$, then we can get (1) by the similar proof of Lemma 3.2 and Theorem 2.2 in Asadian et al. (2006). The details are omitted. □

By the probability inequality (1), we can get the following moment inequality for WOD random variables.

**Lemma 2.3** *Let* $0 < t \leq 2$ *and* $\{X_n, n \geq 1\}$ *be a sequence of WOD random variables. Assume further that* $EX_n = 0$ *for each* $n \geq 1$ *when* $1 \leq t \leq 2$. *Let* $h(x)$ *be a nonnegative even function and nondecreasing on the half-line* $[0, \infty)$. *Assume that* $h(0) = 0$ *and* $Eh(X_i) < \infty$ *for each* $i \geq 1$, *then for every* $r > 0$,

$$Eh(S_n) \leq \sum_{i=1}^{n} Eh(rX_i) + 2g(n)e^r \int_0^{\infty} \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} \mathrm{d}h(x). \quad (2)$$

*Proof* Taking $y = \frac{x}{r}$ in Lemma 2.2, we have

$$P(|S_n| \geq x) \leq \sum_{i=1}^{n} P\left(|X_i| \geq \frac{x}{r}\right) + 2g(n)e^r \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r},$$

which implies that

$$\int_0^{\infty} P(|S_n| \geq x)\,\mathrm{d}h(x) \leq \sum_{i=1}^{n} \int_0^{\infty} P(r|X_i| \geq x)\,\mathrm{d}h(x)$$
$$+2g(n)e^r \int_0^{\infty} \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} \mathrm{d}h(x).$$

Therefore, the desired result (2) follows by the inequality above and Lemma 2.4 in Petrov (1995) immediately. This completes the proof of the lemma. $\square$

By taking $h(x) = |x|^p$, $p \geq t$ in Lemma 2.3, we can get the following moment inequality for WOD random variables.

**Corollary 2.2** *Let $0 < t \leq 2$, $p \geq t$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $E|X_n|^p < \infty$ for each $n \geq 1$. Assume further that $EX_n = 0$ for each $n \geq 1$ when $1 \leq t \leq 2$. Then for any $r > p/t$,*

$$E|S_n|^p \leq r^p M_{p,n} + C(p,t)g(n)M_{t,n}^{p/t}, \tag{3}$$

*where $C(p,t) = 2pe^r t^{-1} B\left(\frac{p}{t}, r - \frac{p}{t}\right) r^{p-p/t}$ depends only on $p$, $t$ and $r$ such that $r > p/t$.*

*Proof* Taking $h(x) = |x|^p$, $p \geq t$ in Lemma 2.3, we can get that for every $r > 0$,

$$E|S_n|^p \leq r^p \sum_{i=1}^{n} E|X_i|^p + 2pg(n)e^r \int_0^\infty x^{p-1}\left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} \mathrm{d}x. \tag{4}$$

It is easy to check that

$$\begin{aligned} I \doteq& \int_0^\infty x^{p-1}\left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} \mathrm{d}x \\ =& \int_0^\infty x^{p-1}\left(\frac{r^{t-1}M_{t,n}}{r^{t-1}M_{t,n} + x^t}\right)^{r} \mathrm{d}x \\ =& \int_0^\infty x^{p-1}\left(1 - \frac{x^t}{r^{t-1}M_{t,n} + x^t}\right)^{r} \mathrm{d}x. \end{aligned}$$

If we set $y = \frac{x^t}{r^{t-1}M_{t,n}+x^t}$ in the last equality above, then we have for $r > p/t$ that

$$I = \frac{r^{p-p/t}M_{t,n}^{p/t}}{t}\int_0^1 y^{\frac{p}{t}-1}(1-y)^{r-\frac{p}{t}-1}\mathrm{d}y = \frac{r^{p-p/t}M_{t,n}^{p/t}}{t} B\left(\frac{p}{t}, r - \frac{p}{t}\right),$$

where

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x, \quad \alpha, \beta > 0$$

is the Beta function. Substitute $I$ to (4), we can obtain the desired result (3) immediately. The proof is completed. □

By Corollary 2.2, we can get the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality for WOD random variables as follows.

**Corollary 2.3** *Let $p \geq 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $E|X_n|^p < \infty$ for each $n \geq 1$. Assume further that $EX_n = 0$ for each $n \geq 1$ when $p \geq 2$. Then there exist positive constants $C_1(p)$ and $C_2(p)$ depending only on $p$ such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq [C_1(p) + C_2(p)g(n)] \sum_{i=1}^{n} E|X_i|^p, \quad for \quad 1 \leq p \leq 2 \tag{5}$$

*and*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C_1(p) \sum_{i=1}^{n} E|X_i|^p + C_2(p)g(n)\left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}, \quad for \quad p \geq 2. \tag{6}$$

*Proof* If $1 \leq p \leq 2$, then (5) follows by (3) immediately by taking $t = p$. If $p \geq 2$, then we can get (6) immediately by taking $t = 2$. The proof is completed. □

The last one is a fundamental inequality for stochastic domination. For the proof, one can refer to Wu (2006) or Shen and Wu (2013).

**Lemma 2.4** *Assume that $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of random variables stochastically dominated by a random variable $X$. Then for all $\alpha > 0$ and $b > 0$, there exist positive constants $C_1$ and $C_2$ such that*

$$E|X_{ni}|^\alpha I\left(|X_{ni}| \leq b\right) \leq C_1\left[E|X|^\alpha I\left(|X| \leq b\right) + b^\alpha P\left(|X| > b\right)\right]$$

*and*

$$E|X_{ni}|^\alpha I\left(|X_{ni}| > b\right) \leq C_2 E|X|^\alpha I\left(|X| > b\right).$$

*Consequently, $E|X_{ni}|^\alpha \leq CE|X|^\alpha$.*

## 3 Complete convergence for arrays of row-wise WOD random variables

In Sect. 2, we get some probability inequalities and moment inequalities for WOD random variables, especially the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality. These inequalities will be applied to prove the complete convergence for weighted sums of arrays of row-wise WOD random variables.

Recently, Kruglov et al. (2006) obtained the following complete convergence theorem for arrays of row-wise independent random variables $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$, where $\{k_n, n \geq 1\}$ is a sequence of positive integers.

**Theorem 3.1** *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of row-wise independent random variables with $EX_{ni} = 0$ for all $1 \leq i \leq k_n, n \geq 1$ and $\{b_n, n \geq 1\}$ be a sequence of nonnegative constants. Suppose that the following conditions hold:*

(i) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$;
(ii) *there exists* $J \geq 1$ *such that*

$$\sum_{n=1}^{\infty} b_n \left( \sum_{i=1}^{k_n} EX_{ni}^2 \right)^J < \infty.$$

*Then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} b_n P\left( \max_{1 \leq m \leq k_n} \left| \sum_{i=1}^{m} X_{ni} \right| > \varepsilon \right) < \infty.$$

Our goal is to extend the result of Theorem 3.1 for arrays of row-wise independent random variables to the case of arrays of row-wise WOD random variables and give its application. One of our main results is as follows.

**Theorem 3.2** *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of row-wise WOD random variables with $EX_{ni} = 0$ for $1 \leq i \leq k_n, n \geq 1$ and $\{b_n, n \geq 1\}$ be a sequence of nonnegative constants. Suppose that the condition $(i)$ of Theorem 3.1 is satisfied and there exist constants $J \geq 1$ and $0 < p \leq 2$ such that*

$$\sum_{n=1}^{\infty} b_n g(k_n) \left( \sum_{i=1}^{k_n} E|X_{ni}|^p \right)^J < \infty. \tag{7}$$

*Then*

$$\sum_{n=1}^{\infty} b_n P\left( \left| \sum_{i=1}^{k_n} X_{ni} \right| > \varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0. \tag{8}$$

*Proof* By Lemma 2.2, we have for $x = \varepsilon$, $y = \varepsilon/J$ and $t = p$ that

$$\begin{aligned} P\left( \left| \sum_{i=1}^{k_n} X_{ni} \right| > \varepsilon \right) &\leq \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon/J) + 2g(k_n)e^J \left( 1 + \frac{\varepsilon^p / J^{p-1}}{\sum_{i=1}^{k_n} E|X_{ni}|^p} \right)^{-J} \\ &\leq \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon/J) + 2g(k_n)e^J J^{J(p-1)} \varepsilon^{-Jp} \left( \sum_{i=1}^{k_n} E|X_{ni}|^p \right)^J, \end{aligned}$$

which implies (8) according to the conditions $(i)$ of Theorem 3.1, (7) and the inequality above immediately. ◻

*Remark 3.1* Taking $g(n) = g_U(n) = g_L(n) = M$, where $M$ is a positive constant, the notion of WOD random variables reduces to END random variables, which contains NOD, NSD, NA and independent random variables as special cases. Hence, the result of Theorem 3.2 still holds for END random variables and the condition (7) can be weakened by

$$\sum_{n=1}^{\infty} b_n \left( \sum_{i=1}^{k_n} E|X_{ni}|^p \right)^J < \infty,$$

where $J \geq 1$ and $0 < p \leq 2$ are constants. Therefore, our result generalizes the corresponding ones for independent random variables, NA random variables, NOD random variables and END random variables. For similar results on complete convergence of NA random variables and NOD random variables, one can refer to Liang (2000), Chen et al. (2008), Qiu et al. (2011), and so forth.

The result of Theorem 3.2 can be applied to establish the following complete convergence result for arrays of row-wise WOD random variables by using the Marcinkiewicz–Zygmund type inequality of WOD random variables. The main idea is inspired by Baek et al. (2008), Wu (2012) and Sung (2012).

**Theorem 3.3** *Suppose that $\beta \geq -1$ and $p \geq 1$. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of row-wise WOD random variables with mean zero, which is stochastically dominated by a random variable $X$. Let $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of constants satisfying*

$$\max_{1 \leq i \leq k_n} |a_{ni}| = O\left(n^{-\gamma}\right) \quad \text{for some} \quad \gamma > 0 \tag{9}$$

*and*

$$\sum_{i=1}^{k_n} |a_{ni}|^q = O\left(n^{-1-\beta+\gamma(p-q)}\right) \quad \text{for some} \quad q < p. \tag{10}$$

*Further assume that*

$$\sum_{i=1}^{k_n} |a_{ni}|^t = O\left(n^{-\alpha}\right) \quad \text{for some} \quad 0 < t \leq 2 \quad \text{and some} \quad \alpha > 0 \tag{11}$$

*if $p \geq 2$. There exists some $0 \leq \lambda < 1$ such that $g(k_n) = O(n^{\gamma\lambda})$, and $0 \leq \lambda < 2 - p$ if $1 < p < 2$. If $E|X|^{p+\lambda} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\beta} P\left( \left| \sum_{i=1}^{k_n} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{12}$$

*Proof* Note that

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni} X_{ni}\right| > \varepsilon\right) \leq \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni}^{+} X_{ni}\right| > \frac{\varepsilon}{2}\right) + \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni}^{-} X_{ni}\right| > \frac{\varepsilon}{2}\right).$$

So, without loss of generality, we assume that $a_{ni} \geq 0$ for all $1 \leq i \leq k_n$ and $n \geq 1$. For $1 \leq i \leq k_n$ and $n \geq 1$, define

$$X_{ni}^{'} = -n^{\gamma} I\left(X_{ni} < -n^{\gamma}\right) + X_{ni} I\left(|X_{ni}| \leq n^{\gamma}\right) + n^{\gamma} I\left(X_{ni} > n^{\gamma}\right),$$
$$X_{ni}^{''} = X_{ni} - X_{ni}^{'} = \left(X_{ni} - n^{\gamma}\right) I\left(X_{ni} > n^{\gamma}\right) + \left(X_{ni} + n^{\gamma}\right) I\left(X_{ni} < -n^{\gamma}\right).$$

By Lemma 2.1 $(i)$, we can see that $\{a_{ni} X_{ni}^{'}, 1 \leq i \leq k_n, n \geq 1\}$ and $\{a_{ni} X_{ni}^{''}, 1 \leq i \leq k_n, n \geq 1\}$ are arrays of row-wise WOD random variables. Since $E X_{ni} = 0$, in order to prove (12), it suffices to show that for all $\varepsilon > 0$,

$$H \doteq \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni}\left(X_{ni}^{'} - E X_{ni}^{'}\right)\right| > \varepsilon\right) < \infty \tag{13}$$

and

$$G \doteq \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni}\left(X_{ni}^{''} - E X_{ni}^{''}\right)\right| > \varepsilon\right) < \infty. \tag{14}$$

We will consider the following three cases.

*Case 1:* $p = 1$.

For $H$, it follows by Markov's inequality and Corollary 2.3 that

$$\begin{aligned} H &\leq C \sum_{n=1}^{\infty} n^{\beta} E\left|\sum_{i=1}^{k_n} a_{ni}\left(X_{ni}^{'} - E X_{ni}^{'}\right)\right|^2 \\ &\leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} a_{ni}^2 E\left|X_{ni}^{'}\right|^2 + C \sum_{n=1}^{\infty} n^{\beta} g(k_n) \sum_{i=1}^{k_n} a_{ni}^2 E\left|X_{ni}^{'}\right|^2 \\ &\doteq H_1 + H_2. \end{aligned} \tag{15}$$

By Lemma 2.4 and conditions (9)–(10), we have

$$H_1 \doteq C \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} a_{ni}^2 E\left|X_{ni}^{'}\right|^2$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty}n^{\beta}\sum_{i=1}^{k_n}a_{ni}^2\left[EX^2I\left(|X|\leq n^{\gamma}\right)+n^{2\gamma}P\left(|X|>n^{\gamma}\right)\right]\\
&\leq C\sum_{n=1}^{\infty}n^{\beta}\max_{1\leq i\leq k_n}|a_{ni}|^{2-q}\sum_{i=1}^{k_n}|a_{ni}|^{q}\left[EX^2I\left(|X|\leq n^{\gamma}\right)+n^{2\gamma}P\left(|X|>n^{\gamma}\right)\right]\\
&\leq C\sum_{n=1}^{\infty}n^{\beta}n^{-\gamma(2-q)}n^{-1-\beta+\gamma(1-q)}\left[EX^2I\left(|X|\leq n^{\gamma}\right)+n^{2\gamma}P\left(|X|>n^{\gamma}\right)\right]\\
&=C\sum_{n=1}^{\infty}n^{-1-\gamma}EX^2I\left(|X|\leq n^{\gamma}\right)+C\sum_{n=1}^{\infty}n^{-1+\gamma}P\left(|X|>n^{\gamma}\right)\\
&=C\sum_{n=1}^{\infty}n^{-1-\gamma}\sum_{i=1}^{n}EX^2I\left((i-1)^{\gamma}<|X|\leq i^{\gamma}\right)\\
&\quad+C\sum_{n=1}^{\infty}n^{-1+\gamma}\sum_{i=n}^{\infty}P\left(i^{\gamma}<|X|\leq (i+1)^{\gamma}\right)\\
&\leq C\sum_{i=1}^{\infty}EX^2I\left((i-1)^{\gamma}<|X|\leq i^{\gamma}\right)i^{-\gamma}+C\sum_{i=1}^{\infty}P\left(i^{\gamma}<|X|\leq (i+1)^{\gamma}\right)i^{\gamma}\\
&\leq CE|X|<\infty. \qquad (16)
\end{aligned}
$$

Similar to the proof of (16), and note that $g(k_n)=O(n^{\gamma\lambda})$ for some $0\leq\lambda<1$, we can see that

$$
\begin{aligned}
H_2&\doteq C\sum_{n=1}^{\infty}n^{\beta}g(k_n)\sum_{i=1}^{k_n}a_{ni}^2E\left|X_{ni}^{'}\right|^2\\
&\leq C\sum_{n=1}^{\infty}n^{-1-\gamma+\gamma\lambda}\sum_{i=1}^{n}EX^2I\left((i-1)^{\gamma}<|X|\leq i^{\gamma}\right)\\
&\quad+C\sum_{n=1}^{\infty}n^{-1+\gamma+\gamma\lambda}\sum_{i=n}^{\infty}P\left(i^{\gamma}<|X|\leq (i+1)^{\gamma}\right)\\
&\leq C\sum_{i=1}^{\infty}EX^2I\left((i-1)^{\gamma}<|X|\leq i^{\gamma}\right)i^{-\gamma+\gamma\lambda}\\
&\quad+C\sum_{i=1}^{\infty}P\left(i^{\gamma}<|X|\leq (i+1)^{\gamma}\right)i^{\gamma+\gamma\lambda}\\
&\leq CE|X|^{1+\lambda}<\infty. \qquad (17)
\end{aligned}
$$

Hence $H<\infty$ follows from (15)–(17) immediately. That is to say, (13) has been proved.

In the following, we will prove (14). Firstly, we prove that

$$\sum_{i=1}^{k_n} |a_{ni}| E \left| X_{ni}^{''} \right| \to 0 \quad \text{as} \quad n \to \infty.$$

Noting that

$$\begin{aligned} \left| X_{ni}^{''} \right| &= \left( X_{ni} - n^{\gamma} \right) I \left( X_{ni} > n^{\gamma} \right) - \left( X_{ni} + n^{\gamma} \right) I \left( X_{ni} < -n^{\gamma} \right) \\ &\le |X_{ni}| I \left( |X_{ni}| > n^{\gamma} \right) \ \le \ |X_{ni}|, \end{aligned}$$

it follows by Lemma 2.4 and conditions (9)–(10) again that

$$\begin{aligned} \sum_{i=1}^{k_n} |a_{ni}| E \left| X_{ni}^{''} \right| &\le \sum_{i=1}^{k_n} |a_{ni}| E |X_{ni}| I \left( |X_{ni}| > n^{\gamma} \right) \\ &\le C \max_{1 \le i \le k_n} |a_{ni}|^{1-q} \sum_{i=1}^{k_n} |a_{ni}|^{q} E |X| I \left( |X| > n^{\gamma} \right) \\ &\le C n^{-1-\beta} E |X| I \left( |X| > n^{\gamma} \right) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence, to prove (14), we only need to show that

$$G^{*} \doteq \sum_{n=1}^{\infty} n^{\beta} P \left( \left| \sum_{i=1}^{k_n} a_{ni} X_{ni}^{''} \right| > \varepsilon \right) < \infty.$$

Putting $0 < \delta < 1$ such that $1 - \delta = p - \delta > q$, we have by Markov's inequality, Lemma 2.4 and conditions (9)–(10) that

$$\begin{aligned} G^{*} &\le C \sum_{n=1}^{\infty} n^{\beta} E \left| \sum_{i=1}^{k_n} a_{ni} X_{ni}^{''} \right|^{1-\delta} \\ &\le C \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} |a_{ni}|^{1-\delta} E |X_{ni}|^{1-\delta} I \left( |X_{ni}| > n^{\gamma} \right) \\ &\le C \sum_{n=1}^{\infty} n^{\beta} \max_{1 \le i \le k_n} |a_{ni}|^{1-\delta-q} \sum_{i=1}^{k_n} |a_{ni}|^{q} E |X|^{1-\delta} I \left( |X| > n^{\gamma} \right) \\ &\le C \sum_{n=1}^{\infty} n^{\beta} n^{-\gamma(1-\delta-q)} n^{-1-\beta+\gamma(1-q)} E |X|^{1-\delta} I \left( |X| > n^{\gamma} \right) \\ &= C \sum_{n=1}^{\infty} n^{-1+\gamma\delta} \sum_{i=n}^{\infty} E |X|^{1-\delta} I \left( i^{\gamma} < |X| \le (i+1)^{\gamma} \right) \end{aligned}$$

$$\leq C\sum_{i=1}^{\infty} E\,|X|^{1-\delta}\, I\left(i^{\gamma} < |X| \leq (i+1)^{\gamma}\right) i^{\gamma\delta}$$
$$\leq CE|X| < \infty.$$

It completes the proof of (14).

*Case 2:* $1 < p < 2$.

In this case, note that $E|X|^p < \infty$ and $g(k_n) = O(n^{\gamma\lambda})$ for some $0 \leq \lambda < 2 - p$. Similar to the proof of Case 1, we have

$$H \leq CE|X|^p + CE|X|^{p+\lambda} < \infty.$$

In the following, we will prove $G < \infty$. Taking $\delta > 0$ such that $p-\delta > \max\{1, q\}$, we have by Markov's inequality, Corollary 2.3 and condition (10) that

$$\begin{aligned}
G &\leq C\sum_{n=1}^{\infty} n^{\beta} E\left|\sum_{i=1}^{k_n} a_{ni}\left(X_{ni}^{''} - EX_{ni}^{''}\right)\right|^{p-\delta} \\
&\leq C\sum_{n=1}^{\infty} n^{\beta} g(k_n)\sum_{i=1}^{k_n} |a_{ni}|^{p-\delta}\, E\left|X_{ni}^{''}\right|^{p-\delta} \text{ (by Corollary 2.3)} \\
&\leq C\sum_{n=1}^{\infty} n^{\beta+\gamma\lambda} \max_{1\leq i\leq k_n} |a_{ni}|^{p-\delta-q}\sum_{i=1}^{k_n} |a_{ni}|^{q}\, E\,|X|^{p-\delta}\, I\left(|X| > n^{\gamma}\right) \\
&\leq C\sum_{n=1}^{\infty} n^{-1+\gamma\delta+\gamma\lambda}\sum_{i=n}^{\infty} E\,|X|^{p-\delta}\, I\left(i^{\gamma} < |X| \leq (i+1)^{\gamma}\right) \\
&\leq C\sum_{i=1}^{\infty} E\,|X|^{p-\delta}\, I\left(i^{\gamma} < |X| \leq (i+1)^{\gamma}\right) i^{\gamma\delta+\gamma\lambda} \\
&\leq CE|X|^{p+\lambda} < \infty. 
\end{aligned} \tag{18}$$

*Case 3:* $p \geq 2$.

In this case, we will show $H < \infty$ and $G < \infty$ by Theorem 3.2.

In order to prove $H < \infty$, taking $\delta > 0$, we have by Markov's inequality, $C_r$ inequality, Lemma 2.4, conditions (9)–(10) that for all $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{k_n} P\left(\left|a_{ni}\left(X_{ni}^{'} - EX_{ni}^{'}\right)\right| > \varepsilon\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{k_n} E\left|a_{ni}\left(X_{ni}^{'} - EX_{ni}^{'}\right)\right|^{p+\delta} \leq C\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{k_n} |a_{ni}|^{p+\delta}\, E\left|X_{ni}^{'}\right|^{p+\delta} \\
&\leq C\sum_{n=1}^{\infty} n^{\beta} \max_{1\leq i\leq k_n} |a_{ni}|^{p+\delta-q}\sum_{i=1}^{k_n} |a_{ni}|^{q}\left[E\,|X|^{p+\delta}\, I\left(|X| \leq n^{\gamma}\right) + n^{\gamma(p+\delta)} P\left(|X| > n^{\gamma}\right)\right]
\end{aligned}$$

$$\leq C\sum_{n=1}^{\infty}n^{-1-\gamma\delta}\sum_{i=1}^{n}E|X|^{p+\delta}I\left((i-1)^{\gamma}<|X|\leq i^{\gamma}\right)$$
$$+C\sum_{n=1}^{\infty}n^{-1+\gamma p}\sum_{i=n}^{\infty}P\left(i^{\gamma}<|X|\leq(i+1)^{\gamma}\right)$$
$$\leq C\sum_{i=1}^{\infty}E|X|^{p+\delta}I\left((i-1)^{\gamma}<|X|\leq i^{\gamma}\right)i^{-\gamma\delta}+C\sum_{i=1}^{\infty}P\left(i^{\gamma}<|X|>(i+1)^{\gamma}\right)i^{\gamma p}$$
$$\leq CE|X|^{p}<\infty. \tag{19}$$

Taking $J\geq 1$ such that $\alpha J-\beta-\gamma\lambda>1$ and noting that $|X'_{ni}|\leq|X_{ni}|$, we have by (11), Lemma 2.4 and $E|X|^{t}<\infty$ (since $0<t\leq 2\leq p$) that

$$\sum_{n=1}^{\infty}n^{\beta}g(k_n)\left(\sum_{i=1}^{k_n}E\left|a_{ni}\left(X'_{ni}-EX'_{ni}\right)\right|^{t}\right)^{J}$$
$$\leq C\sum_{n=1}^{\infty}n^{\beta+\gamma\lambda}\left[\sum_{i=1}^{k_n}|a_{ni}|^{t}\left(E\left|X'_{ni}\right|^{t}+\left(E\left|X'_{ni}\right|\right)^{t}\right)\right]^{J}$$
$$\leq C\sum_{n=1}^{\infty}n^{\beta+\gamma\lambda}\left[\sum_{i=1}^{k_n}|a_{ni}|^{t}\left(E|X|^{t}+(E|X|)^{t}\right)\right]^{J}$$
$$\leq C\sum_{n=1}^{\infty}n^{\beta+\gamma\lambda-\alpha J}<\infty. \tag{20}$$

Therefore, $H<\infty$ follows by Theorem 3.2, (19) and (20) immediately.

In the following, we will prove $G<\infty$. Taking $\delta>0$ such that $p-\delta>\max\{1,q\}$, it follows by the proof of (19) and (18) that

$$\sum_{n=1}^{\infty}n^{\beta}\sum_{i=1}^{k_n}P\left(\left|a_{ni}\left(X''_{ni}-EX''_{ni}\right)\right|>\varepsilon\right)\leq C\sum_{n=1}^{\infty}n^{\beta}\sum_{i=1}^{k_n}E\left|a_{ni}\left(X''_{ni}-EX''_{ni}\right)\right|^{p-\delta}$$
$$\leq C\sum_{n=1}^{\infty}n^{\beta}\sum_{i=1}^{k_n}|a_{ni}|^{p-\delta}E\left|X''_{ni}\right|^{p-\delta}$$
$$\leq CE|X|^{p}<\infty. \tag{21}$$

Noting that $|X''_{ni}|\leq|X_{ni}|$, it follows by the proof of (20) that

$$\sum_{n=1}^{\infty}n^{\beta}g(k_n)\left(\sum_{i=1}^{k_n}E\left|a_{ni}\left(X''_{ni}-EX''_{ni}\right)\right|^{t}\right)^{J}$$
$$\leq C\sum_{n=1}^{\infty}n^{\beta+\gamma\lambda}\left[\sum_{i=1}^{k_n}|a_{ni}|^{t}\left(E|X|^{t}+(E|X|)^{t}\right)\right]^{J}$$
$$\leq C\sum_{n=1}^{\infty}n^{\beta+\gamma\lambda-\alpha J}<\infty, \tag{22}$$

provided that $J \geq 1$ such that $\alpha J - \beta - \gamma\lambda > 1$. Hence, $G < \infty$ follows by Theorem 3.2, (21) and (22) immediately. This completes the proof of the theorem. □

*Remark 3.2* Similar to the proof of Theorem 2 (i) in Sung (2007), we can see that the result of Theorem 3.3 holds for arbitrary arrays of row-wise random variables $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ when $0 < p < 1$ (in this case, the condition mean zero is not needed and $E|X|^{p+\gamma} < \infty$ can be weakened to be $E|X|^p < \infty$).

Combining Theorem 3.3 and Remark 3.2, we can get the following complete convergence for arrays of row-wise WOD random variables.

**Corollary 3.1** *Suppose that $\beta \geq -1$. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of row-wise WOD random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of constants such that (9) holds and*

$$\sum_{i=1}^{k_n} |a_{ni}|^\theta = O(n^\mu) \quad \text{for some} \quad 0 < \theta < 2 \text{ and some } \mu \text{ such that } \theta + \frac{\mu}{\gamma} < 2. \tag{23}$$

*Assume further that there exists some $0 \leq \lambda < 1$ such that $g(k_n) = O(n^{\gamma\lambda})$, and $0 \leq \lambda < 2 - \left(\theta + \frac{1+\mu+\beta}{\gamma}\right)$ if $1 < \theta + \frac{1+\mu+\beta}{\gamma} < 2$, $0 \leq \lambda < -\frac{1+\mu+\beta}{\gamma}$ if $1+\mu+\beta < 0$ and $1 < \theta < 2$.*

(i) *If $1 + \mu + \beta < 0$ and $E|X|^\theta < \infty$, then (12) holds.*
(ii) *If $1 + \mu + \beta > 0$ and*

$$E|X|^s < \infty, \quad \text{where} s = \theta + \frac{1+\mu+\beta}{\gamma} + \lambda,$$

*and assume further that $EX_{ni} = 0$ when $\theta + \frac{1+\mu+\beta}{\gamma} \geq 1$, then (12) hold.*

*Proof* (i) If $1 + \mu + \beta < 0$, we consider the following two cases.

*Case 1:* $0 < \theta \leq 1$.

The result can be easily proved by

$$\begin{aligned}\sum_{n=1}^{\infty} n^\beta P\left(\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| > \varepsilon\right) &\leq C\sum_{n=1}^{\infty} n^\beta E\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right|^\theta \\ &\leq C\sum_{n=1}^{\infty} n^\beta \sum_{i=1}^{k_n} E|a_{ni}X_{ni}|^\theta \\ &\leq C\sum_{n=1}^{\infty} n^{\mu+\beta} E|X|^\theta < \infty.\end{aligned}$$

*Case 2:* $1 < \theta < 2$.

Noting that $0 \le \lambda < -\frac{1+\mu+\beta}{\gamma}$ if $1+\mu+\beta < 0$, we have by Markov's inequality, Corollary 2.3, (23) and $E|X|^{\theta} < \infty$ that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni} X_{ni}\right| > \varepsilon\right) &\le C \sum_{n=1}^{\infty} n^{\beta} E\left|\sum_{i=1}^{k_n} a_{ni} X_{ni}\right|^{\theta} \\ &\le C \sum_{n=1}^{\infty} n^{\beta} g(k_n) \sum_{i=1}^{k_n} E\,|a_{ni} X_{ni}|^{\theta} \\ &\le C \sum_{n=1}^{\infty} n^{\mu+\beta+\gamma\lambda} E|X|^{\theta} \ < \ \infty.\end{aligned}$$

(ii) If $1+\mu+\beta > 0$, we will apply Theorem 3.3 with $p = \theta + \frac{1+\mu+\beta}{\gamma}$ and $q = \theta$. By (9) and (23), we can see that (10) holds and

$$\sum_{i=1}^{k_n} a_{ni}^2 \le \max_{1 \le i \le k_n} |a_{ni}|^{2-\theta} \sum_{i=1}^{k_n} |a_{ni}|^{\theta} = O\left(n^{-(\gamma(2-\theta)-\mu)}\right) \doteq O(n^{-\alpha}),$$

where $\alpha = \gamma(2-\theta) - \mu > 0$. That is to say (11) holds for $t = 2$. Thus, (12) follows by Theorem 3.3 immediately. The proof is complete. □

By using Corollary 3.1, we can get the following result for WOD random variables.

**Corollary 3.2** *Let $p \ge 1$, $0 < \alpha < 2$ and $p\alpha > 1$. Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of row-wise WOD random variables with mean zero, which is stochastically dominated by a random variable $X$. Assume further that there exists some $0 \le \lambda < 1$ such that $g(n) = O(n^{\lambda/\alpha})$, and $0 \le \lambda < 2 - p\alpha$ if $1 < p\alpha < 2$. Then $E|X|^{p\alpha+\lambda}$ implies that*

$$\sum_{n=1}^{\infty} n^{p-2} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon n^{1/\alpha}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{24}$$

*Proof* Let $a_{ni} = 0$ if $i > n$ and $a_{ni} = n^{-1/\alpha}$ if $1 \le i \le n$. Hence, conditions (9) and (23) hold for $\theta = 1$, $\gamma = 1/\alpha$ and $\mu = 1 - 1/\alpha$ such that $\theta + \frac{\mu}{\gamma} = \alpha < 2$, where $k_n = n$, $\beta \doteq p - 2 \ge -1$. It can be found that

$$1 + \mu + \beta = p - 1/\alpha > 0, \quad s \doteq \theta + \frac{1+\mu+\beta}{\gamma} + \lambda = p\alpha + \lambda.$$

Hence, the desired result (24) follows by Corollary 3.1 $(ii)$ immediately. The proof is complete. □

## 4 Complete consistency for the estimator of nonparametric regression model based on WOD errors

In this section, we will give some applications of complete convergence obtained in Sect. 3 in nonparametric regression models based on WOD errors. The complete consistency for the estimator of nonparametric regression model will be investigated in this section.

Consider the following nonparametric regression model:

$$Y_{ni} = f(x_{ni}) + \varepsilon_{ni}, \quad i = 1, 2, \ldots, n, \tag{25}$$

where $x_{ni}$ are known fixed design points from $A$, $f(\cdot)$ is an unknown regression function defined on $A$ and $\varepsilon_{ni}$ are random errors. Here and below, $A \subset \mathrm{R}^d$ is a given compact set for some positive integer $d \geq 1$. As an estimator of $f(\cdot)$, we will consider the following weighted regression estimator:

$$f_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_{ni}, \quad x \in A \subset \mathrm{R}^d, \tag{26}$$

where $W_{ni}(x) = W_{ni}(x; x_{n1}, x_{n2}, \ldots, x_{nn})$, $i = 1, 2, \ldots, n$ are the weight functions.

This class of estimator (26) was first introduced by Stone (1977) and next adapted by Georgiev (1983) to the fixed design case. Up to now, the estimator (26) has been studied by many authors, especially the strong consistency, mean consistency, complete consistency and the asymptotic normality for independent errors or dependent errors. For more details about the consistency or asymptotic normality, one can refer to Roussas (1989), Fan (1990), Roussas et al. (1992), Tran et al. (1996), Hu et al. (2002), Liang and Jing (2005), Yang et al. (2012), and so forth. The main purpose of this section is to further investigate the complete consistency of the estimator $f_n(x)$ under WOD errors by using the complete convergence obtained in Sect. 3. The proofs for the main results here are different from other literatures that studied the consistency of the estimator $f_n(x)$ of nonparametric regression model.

### 4.1 Theoretical results

In this subsection, let $c(f)$ denote all continuity points of the function $f$ on $A$. The symbol $\|x\|$ denotes the Euclidean norm. For any point $x \in A$, we will consider the following assumptions on weight functions $W_{ni}(x)$:

$(H_1)$ $\sum_{i=1}^{n} W_{ni}(x) \to 1$ as $n \to \infty$;
$(H_2)$ $\sum_{i=1}^{n} |W_{ni}(x)| \leq C < \infty$ for all $n$;
$(H_3)$ $\sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

Based on the assumptions above, we will further study the complete consistency of the nonparametric regression estimator $f_n(x)$ by using Corollary 3.1 obtained in

Sect. 3. Our main results on complete consistency of the nonparametric regression estimator $f_n(x)$ are as follows.

**Theorem 4.1** *Let* $\{\varepsilon_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of row-wise WOD random variables with mean zero, which is stochastically dominated by a random variable* $X$. *Suppose that the conditions* $(H_1)$*–*$(H_3)$ *hold true, and*

$$\max_{1\le i\le n} |W_{ni}(x)| = O\left(n^{-\gamma}\right) \quad \textit{for some} \quad 0 < \gamma \le 1.$$

*Assume further that there exists some* $0 \le \lambda < 1$ *such that* $g(n) = O(n^{\gamma\lambda})$. *If* $E|X|^{\frac{2}{\gamma}+\lambda} < \infty$, *then for any* $x \in c(f)$,

$$f_n(x) \to f(x) \quad \textit{completely, as} \quad n \to \infty. \tag{27}$$

*Proof* For $x \in c(f)$ and $a > 0$, it follows by (25) and (26) that

$$\begin{aligned} |Ef_n(x) - f(x)| \le & \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)|\, I(\|x_{ni} - x\| \le a) \\ & + \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)|\, I(\|x_{ni} - x\| > a) \\ & + |f(x)| \cdot \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right|. \end{aligned} \tag{28}$$

Since $x \in c(f)$, it follows that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x^*) - f(x)| < \varepsilon$ when $\|x^* - x\| < \delta$. Taking $a \in (0, \delta)$ in (28), we have

$$\begin{aligned} |Ef_n(x) - f(x)| \le & \ \varepsilon \sum_{i=1}^{n} |W_{ni}(x)| \ + \ |f(x)| \cdot \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right| \\ & + \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)|\, I(\|x_{ni} - x\| > a). \end{aligned} \tag{29}$$

It follows by (29) and conditions $(H_1)$–$(H_3)$ that for any $x \in c(f)$,

$$\lim_{n\to\infty} Ef_n(x) = f(x).$$

Hence, to prove (27), it suffices to show that

$$f_n(x) - Ef_n(x) = \sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni} \to 0 \quad \text{completely,} \quad \text{as} \quad n \to \infty,$$

that is to say,

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}\right| > \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{30}$$

Note that for any point $x \in c(f)$,

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}\right| > \varepsilon\right) \leq \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}^{+}(x)\varepsilon_{ni}\right| > \frac{\varepsilon}{2}\right) + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}^{-}(x)\varepsilon_{ni}\right| > \frac{\varepsilon}{2}\right).$$

So, without loss of generality, we assume that $W_{ni}(x) > 0$. Applying Corollary 3.1 with $\mu = 0, \theta = 1$ and $\beta = 1 - \gamma \geq 0$, we have $1 + \mu + \beta = 2 - \gamma > 0$ and $s = \theta + \frac{1+\mu+\beta}{\gamma} + \lambda = \frac{2}{\gamma} + \lambda$. Denote $a_{ni} = W_{ni}(x)$ in Corollary 3.1. It can be found that the conditions (9) and (23) in Corollary 3.1 are satisfied. Hence, it follows by Corollary 3.1 $(ii)$ that

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}\right| > \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > 0,$$

which implies (30). The proof is complete. □

**Theorem 4.2** *Let $\{\varepsilon_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of row-wise WOD random variables with mean zero, which is stochastically dominated by a random variable $X$. Suppose that the conditions $(H_1)$–$(H_3)$ hold true, and*

$$\max_{1\leq i\leq n} |W_{ni}(x)| = O\left(n^{-\gamma}\right) \quad \textit{for some} \quad \gamma > 0.$$

*Assume further that there exists some $0 \leq \lambda < 1$ such that $g(n) = O(n^{\gamma\lambda})$, and $0 \leq \lambda < 1 - \frac{1}{\gamma}$ if $\gamma > 1$. If $E|X|^{1+1/\gamma+\lambda} < \infty$, then (27) holds.*

*Proof* The proof is similar to that of Theorem 4.1. Under the conditions of Theorem 4.2, we only need to prove (30). We will apply Corollary 3.1 with $\mu = 0, \theta = 1$ and $\beta = 0$. Hence, $1 + \mu + \beta = 1 > 0$ and $s = \theta + \frac{1+\mu+\beta}{\gamma} + \lambda = 1 + \frac{1}{\gamma} + \lambda$. Denote $a_{ni} = W_{ni}(x)$ in Corollary 3.1. It can be found that the conditions (9) and (23) in Corollary 3.1 are satisfied. Hence, the desired result (30) follows by Corollary 3.1 $(ii)$ immediately. The proof is complete. □

**Theorem 4.3** *Let $p > 1$ and $\{\varepsilon_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of row-wise WOD random variables with mean zero, which is stochastically dominated by a random variable $X$. Suppose that the conditions $(H_1)$–$(H_3)$ hold true, and*

$$\max_{1\leq i\leq n} |W_{ni}(x)| = O\left(n^{-\gamma}\right) \quad \textit{for some} \quad \gamma \geq \frac{1}{p-1}.$$

*Assume further that there exists some* $0 \leq \lambda < 1$ *such that* $g(n) = O(n^{\gamma\lambda})$*, and* $0 \leq \lambda < 2 - p$ *if* $1 < p < 2$*. If* $E|X|^{p+\lambda} < \infty$*, then* (27) *holds.*

*Proof* The proof is similar to that of Theorem 4.1. Under the conditions of Theorem 4.3, we still only need to prove (30). We will apply Corollary 3.1 with $\mu = 0, \theta = 1$ and $\beta = \gamma(p-1) - 1 \geq 0$. Hence, $1 + \mu + \beta = \gamma(p-1) > 0$ and $s = \theta + \frac{1+\mu+\beta}{\gamma} + \lambda = p + \lambda$. Denote $a_{ni} = W_{ni}(x)$ in Corollary 3.1. It can be found that the conditions (9) and (23) in Corollary 3.1 are satisfied. Hence, the desired result (30) follows by Corollary 3.1 $(ii)$ immediately. The proof is complete. □

*Remark 4.1* If $g(n) = g_U(n) = g_L(n) = M$, where $M$ is a positive constant, then $g(n) = O(n^{\gamma\lambda})$ holds for $\lambda = 0$. By using Theorems 4.1–4.3, we can get the similar results on complete consistency of the nonparametric regression estimator $f_n(x)$ under END errors, which contains NOD, NSD, NA and independent random variables as special cases.

*Remark 4.2* In Yang et al. (2012), they studied the mean convergence and almost sure convergence of the nonparametric regression estimator $f_n(x)$ under NOD errors. In this paper, we further investigate the complete convergence of the nonparametric regression estimator $f_n(x)$ under WOD errors, which contains NOD as a special case. Hence, our main results on complete consistency of the nonparametric regression estimator $f_n(x)$ also hold under NOD errors.

*Remark 4.3* Theorem 4.2 under NA errors is considered by Liang and Jing (2005), so the result of Theorem 4.2 generalizes the corresponding one of Liang and Jing (2005) under NA errors to the case of WOD errors. We point out that the method used to prove Theorem 4.2 is different from that of Liang and Jing (2005). In addition, Theorems 4.1 and 4.3, which provide some different conditions from Theorem 4.2, are not considered by Liang and Jing (2005). Since WOD contains NA as a special case, Theorems 4.1 and 4.3 also hold under NA errors.

*Remark 4.4* Roussas (1989) discussed strong consistency and quadratic mean consistency of $f_n(x)$, and Roussas et al. (1992) established asymptotic normality of $f_n(x)$ assuming that the errors form a strictly stationary stochastic process and satisfying the strong mixing condition, while this paper is devoted to establish the complete consistency of the nonparametric regression estimator $f_n(x)$ under WOD errors.

### 4.2 The choice of the fixed design points and the weight functions

In this subsection we show that the designed assumptions $(H_1)$–$(H_3)$ are satisfied for nearest neighbor weights. For simplicity, we assume that $A = [0, 1]$, taking $x_{ni} = \frac{i}{n}$, $i = 1, 2, \ldots, n$. For any $x \in A$, we rewrite $|x_{n1} - x|, |x_{n2} - x|, \ldots, |x_{nn} - x|$ as follows:

$$\left|x^{(n)}_{R_1(x)} - x\right| \leq \left|x^{(n)}_{R_2(x)} - x\right| \leq \cdots \leq \left|x^{(n)}_{R_n(x)} - x\right|,$$

if $|x_{ni} - x| = \left|x_{nj} - x\right|$, then $|x_{ni} - x|$ is permuted before $\left|x_{nj} - x\right|$ when $x_{ni} < x_{nj}$.

Let $1 \le k_n \le n$, the nearest neighbor weight function estimator of $f(x)$ in model (25) is defined as follows:

$$\tilde{f}_n(x) = \sum_{i=1}^{n} \tilde{W}_{ni}(x) Y_{ni},$$

where

$$\tilde{W}_{ni}(x) = \begin{cases} 1/k_n, & \text{if } |x_{ni} - x| \le \left| x^{(n)}_{R_{k_n}(x)} - x \right|, \\ 0, & \text{otherwise.} \end{cases}$$

Assume further that $f$ is continuous on the compact set $A$. It is easily checked that for any $x \in [0, 1]$, if follows by the definitions of $R_i(x)$ and $\tilde{W}_{ni}(x)$ that

$$\sum_{i=1}^{n} \tilde{W}_{ni}(x) = \sum_{i=1}^{n} \tilde{W}_{nR_i(x)}(x) = \sum_{i=1}^{k_n} \frac{1}{k_n} = 1,$$
$$\max_{1 \le i \le n} \tilde{W}_{ni}(x) = \frac{1}{k_n}, \quad \tilde{W}_{ni}(x) \ge 0,$$

and

$$\begin{aligned} &\sum_{i=1}^{n} \left| \tilde{W}_{ni}(x) \right| \cdot |f(x_{ni}) - f(x)| \, I(|x_{ni} - x| > a) \\ &\quad \le C \sum_{i=1}^{n} \frac{(x_{ni} - x)^2 \left| \tilde{W}_{ni}(x) \right|}{a^2} \\ &\quad = C \sum_{i=1}^{k_n} \frac{\left( x^{(n)}_{R_i(x)} - x \right)^2}{k_n a^2} \le C \sum_{i=1}^{k_n} \frac{\left(\frac{i}{n}\right)^2}{k_n a^2} \\ &\quad \le C \left( \frac{k_n}{na} \right)^2, \quad \forall a > 0. \end{aligned}$$

If we take $k_n = \lceil n^s \rceil$ for some $0 < s < 1$, then the conditions $(H_1)$–$(H_3)$ are satisfied.

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