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MOMENT INEQUALITIES FOR m-NOD RANDOM VARIABLES AND THEIR APPLICATIONS*

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Abstract. The concept of m-negatively orthant dependent (m-NOD) random variables is introduced, and the moment inequalities for m-NOD random variables, especially the Marcinkiewicz–Zygmund-type inequality and Rosenthal-type inequality, are established. As one application of the moment inequalities, we study the L_r convergence and strong convergence for m-NOD random variables under some uniformly integrable conditions. On the other hand, the asymptotic approximation of inverse moments for nonnegative m-NOD random variables with finite first moments is established. The results obtained in the paper generalize or improve some known ones for independent sequences and some dependent sequences.

Key words. m-negatively orthant dependent sequence, L_r-convergence, inverse moments, Marcinkiewicz–Zygmund-type inequalities, Rosenthal inequality

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1. Introduction.

It is well known that the Marcinkiewicz–Zygmund-type inequality and Rosenthal-type inequality play important roles in probability limit theory and mathematical statistics, especially in establishing strong convergence, complete convergence, weak convergence, consistency, and asymptotic normality in many stochastic models. There are many sequences of random variables satisfying the Marcinkiewicz–Zygmund-type inequality or the Rosenthal-type inequality under some suitable conditions such as an independent sequence, a φ-mixing sequence with the mixing coefficients satisfying certain conditions (see [36]), a ρ-mixing sequence with the mixing coefficients satisfying certain conditions (see [20]), a $\widetilde{\rho}$-mixing sequence (see [32] or [40]), a negatively associated (NA) sequence (see [21]), a negatively orthant dependent (NOD) sequence (see [2]), an extended negatively dependent(END) sequence (see [22]), a negatively superadditive dependent (NSD) sequence (see [11] or [39]), an asymptotically almost negatively associated (AANA) sequence with the mixing coefficients satisfying certain conditions (see [49]), a ρ^--mixing sequence with the mixing coefficients satisfying certain conditions (see [34]), and so on.

The main purpose of the paper is to introduce the new concept of dependent structure—m-negatively orthant dependence (m-NOD)—and establish the Marcinkiewicz–Zygmund-type inequality and the Rosenthal-type inequality for m-NOD random variables. In addition, we will give some applications of the Marcinkiewicz–Zygmund-type inequality and the Rosenthal-type inequality to L_r-convergence, strong law of large numbers, and the asymptotic approximation of inverse moments for nonnegative m-NOD random variables with finite first moments.

First, let us recall the definition of NOD random variables, which was introduced by Joag-Dev and Proschan [14] as follows.

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DEFINITION 1.1. *A finite collection of random variables* $X_1, \dots, X_n$ *is said to be* negatively orthant dependent (NOD) *if*

$$\mathbf{P}(X_1 > x_1, \dots, X_n > x_n) \leqslant \prod_{i=1}^{n} \mathbf{P}(X_i > x_i)$$

and

$$\mathbf{P}(X_1 \leqslant x_1, \dots, X_n \leqslant x_n) \leqslant \prod_{i=1}^{n} \mathbf{P}(X_i \leqslant x_i)$$

for each $n \geqslant 1$ *and all real numbers* $x_1, \dots, x_n$. *An infinite sequence* $\{X_n,\, n \geqslant 1\}$ *is said to be NOD if every finite subcollection is NOD.*

An array $\{X_{ni},\, i \geqslant 1,\, n \geqslant 1\}$ *of random variables is said to be rowwise NOD if, for every* $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ *is a sequence of NOD random variables.*

The class of NOD random variables is a very general dependent structure, which includes independent random variables and NA random variables as special cases. For more details about the probability inequalities, moment inequalities, or probability limit theory and applications, the reader can refer to [14], [4], [33], [31], [16], [2], [41], [37], [38], [42], [50], [19], [25], [45], etc.

Inspired by the definition of NOD random variables, we introduce the concept of m-NOD random variables as follows.

DEFINITION 1.2. *Let* $m \geqslant 1$ *be a fixed integer. A sequence* $\{X_n, n \geqslant 1\}$ *of random variables is said to be* m-negatively orthant dependent (m-NOD) *if, for any* $n \geqslant 2$ *and any* $i_1, \dots, i_n$ *such that* $|i_k - i_j| \geqslant m$ *for all* $1 \leqslant k \neq j \leqslant n$, *we have that* $X_{i_1}, \dots, X_{i_n}$ *are NOD.*

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ *of random variables is said to be rowwise m-NOD if, for every* $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ *is a sequence of m-NOD random variables.*

For $n = 2$, m-NOD reduces to m-pairwise NOD, which was introduced by Anh in [1] and carefully studied by Wu and Rosalsky in [46]. For $m = 1$, the concept m-NOD random variables reduces to the so-called NOD random variables. Hence, the concept of m-NOD random variables is a natural extension from NOD random variables. Joag-Dev and Proschan in [14]) pointed out that NA implies NOD, but NOD does not implies NA. Hu and Yang [12] (see also Hu, Xie, and Ruan [13] pointed out that NSD implies NOD. Hence, the class of m-NOD random variables includes independent random variables, NA random variables, NSD random variables, NOD random variables, and m-NA random variables (see [10]) as special cases. Studying the probability inequalities, moment inequalities, and limiting behavior of m-NOD random variables and their applications in many stochastic models is of great interest.

The following lemmas for NOD random variables will be used in establishing the Marcinkiewicz–Zygmund-type inequality and the Rosenthal-type inequality for m-NOD random variables.

LEMMA 1.1 (cf. [4]). *Let random variables* $X_1, \dots, X_n$ *be NOD,* $f_1, \dots, f_n$ *be all nondecreasing* (*or all nonincreasing*) *functions*; *then random variables* $f_1(X_1), \dots, f_n(X_n)$ *are NOD.*

LEMMA 1.2 (cf. [2]). *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of NOD random variables with* $\mathbf{E}X_n = 0$ *and* $\mathbf{E}|X_n|^p < \infty$ *for some* $p \geqslant 1$ *and every* $n \geqslant 1$. *Then there exist* *positive constants C_p and D_p depending only on p such that, for every $n \geqslant 1$,*

$$\mathbf{E}\Big|\sum_{i=1}^{n} X_i\Big|^p \leqslant C_p \sum_{i=1}^{n} \mathbf{E}|X_i|^p \quad \textit{for } 1 \leqslant p \leqslant 2 \tag{1.1}$$

and

$$\mathbf{E}\Big|\sum_{i=1}^{n} X_i\Big|^p \leqslant D_p\bigg\{\sum_{i=1}^{n} \mathbf{E}|X_i|^p + \bigg(\sum_{i=1}^{n} \mathbf{E}X_i^2\bigg)^{p/2}\bigg\} \quad \textit{for } p > 2. \tag{1.2}$$

Throughout the paper, let C denote a positive constant independent of n, which may be different in various places; $a_n = O(b_n)$ stands for $a_n \leqslant Cb_n$, where $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ are sequences of nonnegative real numbers. Denote $\log x = \ln \max(x, e)$, $x^+ = x\mathbf{I}(x > 0)$, $x^- = -x\mathbf{I}(x < 0)$.

This work is organized as follows: the Marcinkiewicz–Zygmund-type inequality and the Rosenthal-type inequality for m-NOD random variables are provided in section 2. Some results on L_r-convergence and strong law of large numbers for arrays of rowwise m-NOD random variables are established in section 3. The asymptotic approximation of inverse moments for nonnegative m-NOD random variables with finite first moments is investigated in section 4.

2. Marcinkiewicz–Zygmund-type inequality and the Rosenthal-type inequality for m-NOD random variables.

In this section, we will establish the Marcinkiewicz–Zygmund-type inequality and the Rosenthal-type inequality for m-NOD random variables, which can be applied to prove the strong convergence, L_r-convergence, weak convergence, complete convergence, consistency, and asymptotic normality in many stochastic models, and so on. To prove the main results, we need the following lemma, which will be used frequently throughout the paper.

LEMMA 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-NOD random variables. If $\{f_n(\cdot), n \geqslant 1\}$ are all nondecreasing (or nonincreasing) functions, then the random variables $\{f_n(X_n), n \geqslant 1\}$ are m-NOD.*

This lemma can be easily obtained from the definition of m-NOD random variables and Lemma 1.1. So we omit the details.

Using Lemmas 1.2 and 2.1 we can establish the Marcinkiewicz–Zygmund type inequality and the Rosenthal-type inequality for m-NOD random variables as follows.

THEOREM 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-NOD random variables with $\mathbf{E}X_n = 0$ and $\mathbf{E}|X_n|^p < \infty$ for some $p \geqslant 1$ and every $n \geqslant 1$. Then there exist positive constants $C_{m,p}$ and $D_{m,p}$ depending only on m and p such that, for every $n \geqslant m$,*

$$\mathbf{E}\Big|\sum_{i=1}^{n} X_i\Big|^p \leqslant \begin{cases} C_{m,p}\displaystyle\sum_{i=1}^{n} \mathbf{E}|X_i|^p & \textit{for } 1 \leqslant p \leqslant 2, \\ D_{m,p}\bigg[\displaystyle\sum_{i=1}^{n} \mathbf{E}|X_i|^p + \bigg(\sum_{i=1}^{n} \mathbf{E}X_i^2\bigg)^{p/2}\bigg] & \textit{for } p > 2 \end{cases} \tag{2.1}$$

and

$$\mathbf{E}\biggl(\max_{1\leqslant j\leqslant n}\biggl|\sum_{i=1}^{j}X_i\biggr|^p\biggr)\leqslant\begin{cases}C_{m,p}\ln^p n\sum_{i=1}^{n}\mathbf{E}|X_i|^p & \text{for } 1\leqslant p\leqslant 2,\\ D_{m,p}\ln^p n\biggl[\sum_{i=1}^{n}\mathbf{E}|X_i|^p+\biggl(\sum_{i=1}^{n}\mathbf{E}X_i^2\biggr)^{p/2}\biggr] & \\ \qquad\qquad \text{for } p>2.\end{cases}\tag{2.2}$$

Proof. From (2.1) we see that (2.2) can be immediately obtained similarly as Theorem 2.3.1 from [28]. So we only need to prove (2.1).

For fixed $n\geqslant m$, let $r=\lceil n/m\rceil$. Define

$$Y_i=\begin{cases}X_i, & 1\leqslant i\leqslant n,\\ 0, & i>n.\end{cases}$$

Denote $S'_{mr+j}=\sum_{i=0}^{r}Y_{mi+j}$ for $j=1,\ldots,m$. Noting that $\sum_{i=1}^{n}X_i=\sum_{j=1}^{m}S'_{mr+j}$, we have by the C_r-inequality that

$$\mathbf{E}\biggl|\sum_{i=1}^{n}X_i\biggr|^p=\mathbf{E}\biggl|\sum_{j=1}^{m}S'_{mr+j}\biggr|^p\leqslant m^{p-1}\sum_{j=1}^{m}\mathbf{E}|S'_{mr+j}|^p.\tag{2.3}$$

By definition of m-NOD random variables, we see that $Y_j,Y_{m+j},\ldots,Y_{mr+j}$ are NOD random variables for each $j=1,\ldots,m$.

For $1\leqslant p\leqslant 2$, it follows from (1.1) and (2.3) that, for any $n\geqslant m$,

$$\begin{aligned}\mathbf{E}\biggl|\sum_{i=1}^{n}X_i\biggr|^p&\leqslant m^{p-1}C_p\sum_{j=1}^{m}\sum_{i=0}^{r}\mathbf{E}|Y_{mi+j}|^p\\ &\leqslant m^pC_p\sum_{i=1}^{n}\mathbf{E}|X_i|^p\doteq C_{m,p}\sum_{i=1}^{n}\mathbf{E}|X_i|^p.\end{aligned}\tag{2.4}$$

For $p>2$, it follows from (1.2) and (2.3) that, for any $n\geqslant m$,

$$\begin{aligned}\mathbf{E}\biggl|\sum_{i=1}^{n}X_i\biggr|^p&\leqslant m^{p-1}D_p\sum_{j=1}^{m}\biggl[\sum_{i=0}^{r}\mathbf{E}|Y_{mi+j}|^p+\biggl(\sum_{i=0}^{r}\mathbf{E}Y_{mi+j}^2\biggr)^{p/2}\biggr]\\ &\leqslant m^pD_p\biggl[\sum_{i=1}^{n}\mathbf{E}|X_i|^p+\biggl(\sum_{i=1}^{n}\mathbf{E}X_i^2\biggr)^{p/2}\biggr]\\ &\doteq D_{m,p}\biggl[\sum_{i=1}^{n}\mathbf{E}|X_i|^p+\biggl(\sum_{i=1}^{n}\mathbf{E}X_i^2\biggr)^{p/2}\biggr].\end{aligned}\tag{2.5}$$

Now the desired result (2.1) readily follows by (2.4) and (2.5). This completes the proof of the theorem.

Remark 2.1. Assume that (2.1) holds for any $n\geqslant m$ and the series $\sum_{i=1}^{\infty}X_i$ converges almost surely. Then

$$\mathbf{E}\biggl|\sum_{i=1}^{\infty}X_i\biggr|^p\leqslant\begin{cases}C_{m,p}\sum_{i=1}^{\infty}\mathbf{E}|X_i|^p & \text{for } 1\leqslant p\leqslant 2,\\ D_{m,p}\biggl[\sum_{i=1}^{\infty}\mathbf{E}|X_i|^p+\biggl(\sum_{i=1}^{\infty}\mathbf{E}X_i^2\biggr)^{p/2}\biggr] & \text{for } p>2.\end{cases}\tag{2.6}$$

In fact, using Fatou's lemma, this gives

$$\mathbf{E}\Big|\sum_{i=1}^{\infty} X_i\Big|^p = \mathbf{E}\Big|\liminf_{n\to\infty}\sum_{i=1}^{n} X_i\Big|^p \leqslant \mathbf{E}\Big(\liminf_{n\to\infty}\Big|\sum_{i=1}^{n} X_i\Big|^p\Big)$$
$$\leqslant \liminf_{n\to\infty}\mathbf{E}\Big|\sum_{i=1}^{n} X_i\Big|^p \leqslant \limsup_{n\to\infty}\mathbf{E}\Big|\sum_{i=1}^{n} X_i\Big|^p, \tag{2.7}$$

which, together with (2.1), yields (2.6).

Remark 2.2. Let $\{a_n, n \geqslant 1\}$ be a sequence of real numbers. Under the conditions of Theorem 2.1, we have, for $n \geqslant m$,

$$\mathbf{E}\Big|\sum_{i=1}^{n} a_i X_i\Big|^p \leqslant \begin{cases} 2^{p-1} C_{m,p}\sum_{i=1}^{n}\mathbf{E}|a_i X_i|^p & \text{for } 1 \leqslant p \leqslant 2, \\ 2^p D_{m,p}\Big[\sum_{i=1}^{n}\mathbf{E}|a_i X_i|^p + \Big(\sum_{i=1}^{n}\mathbf{E}a_i^2 X_i^2\Big)^{p/2}\Big] & \\ & \text{for } p > 2. \end{cases} \tag{2.8}$$

Assume further that $\sum_{i=1}^{\infty} a_i X_i$ converges almost surely. Then, for $n \geqslant m$,

$$\mathbf{E}\Big|\sum_{i=1}^{\infty} a_i X_i\Big|^p \leqslant \begin{cases} 2^{p-1} C_{m,p}\sum_{i=1}^{\infty}\mathbf{E}|a_i X_i|^p & \text{for } 1 \leqslant p \leqslant 2, \\ 2^p D_{m,p}\Big[\sum_{i=1}^{\infty}\mathbf{E}|a_i X_i|^p + \Big(\sum_{i=1}^{\infty}\mathbf{E}a_i^2 X_i^2\Big)^{p/2}\Big] & \\ & \text{for } p > 2. \end{cases} \tag{2.9}$$

Actually, for fixed $n \geqslant m$, $\{a_i^+ X_i, 1 \leqslant i \leqslant n\}$ and $\{a_i^- X_i, 1 \leqslant i \leqslant n\}$ are both m-NOD random variables from Lemma 2.1. Noting that $a_{ni} = a_{ni}^+ - a_{ni}^-$, it follows from the C_r-inequality that

$$\mathbf{E}\Big|\sum_{i=1}^{n} a_i X_i\Big|^p \leqslant 2^{p-1}\mathbf{E}\Big|\sum_{i=1}^{n} a_i^+ X_i\Big|^p + 2^{p-1}\mathbf{E}\Big|\sum_{i=1}^{n} a_i^- X_i\Big|^p. \tag{2.10}$$

Noting that $|a_i|^p = (a_i^+)^p + (a_i^-)^p$, the desired result (2.8) readily follows from (2.1) and (2.10).

Estimate (2.9) easily follows from (2.8) as in the proof of (2.7).

3. L_r-convergence and strong convergence for m-NOD random variables. In the previous section, we established the Marcinkiewicz–Zygmund-type inequality and the Rosenthal-type inequality for m-NOD random variables. As one application of the moment inequalities for m-NOD random variables, we will study the L_r-convergence and strong convergence for m-NOD random variables under some uniformly integrable conditions.

In what follows, let $\{u_n, n \geqslant 1\}$ and $\{v_n, n \geqslant 1\}$ be two sequences of integers (not necessarily positive or finite) such that $v_n > u_n$ for all $n \geqslant 1$ and $v_n - u_n \to \infty$ as $n \to \infty$. Let $\{k_n, n \geqslant 1\}$ be a sequence of positive numbers such that $k_n \to \infty$ as $n \to \infty$, and let $\{h(n), n \geqslant 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$.

3.1. L_r-convergence and weak law of large numbers. The notion of h-integrability for an array of random variables concerning an array of constant weights was introduced by Cabrera and Volodin [17] as follows.

Definition 3.1. *Let $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of random variables, and let $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of constants with $\sum_{i=u_n}^{v_n} |a_{ni}| \leqslant C$ for all $n \in \mathbf{N}$ and some constant $C > 0$. The array $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ is said to be h-*integrable with respect to the array of constants $\{a_{ni}\}$ *if*

$$\sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|\mathbf{E}|X_{ni}| < \infty \quad \text{and} \quad \lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{ni}|\mathbf{E}|X_{ni}|\mathbf{I}(|X_{ni}| > h(n)) = 0.$$

The main idea of the notion of h-integrability with respect to the array of constants $\{a_{ni}\}$ is to deal with weighted sums of random variables. Sung, Lisawadi, and Volodin [29] introduced a new concept of integrability which deals with usual normed sums of random variables as follows.

Definition 3.2. *Let $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of random variables and $r > 0$. The array $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ is said to be h-*integrable with exponent r *if*

$$\sup_{n\geqslant 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbf{E}|X_{ni}|^r < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbf{E}|X_{ni}|^r\mathbf{I}(|X_{ni}|^r > h(n)) = 0.$$

Under the conditions of h-integrability with exponent r and h-integrability with respect to the array of constants $\{a_{ni}\}$, Sung, Lisawadi, and Volodin [29] obtained the following Theorem A and Theorem B for arrays of rowwise NA random variables, respectively.

Theorem A. *Let $1 \leqslant r < 2$. Suppose that $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ is an array of rowwise NA random variables. Let $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of constants. Assume that the following conditions hold*:

(i) *$\{|X_{ni}|^r\}$ is h-integrable concerning the array $\{|a_{ni}|^r\}$, i.e.,*

$$\sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r\mathbf{E}|X_{ni}|^r < \infty$$

and

$$\lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r\mathbf{E}|X_{ni}|^r\mathbf{I}(|X_{ni}|^r > h(n)) = 0;$$

(ii) *$h(n)\sup_{u_n\leqslant i\leqslant v_n} |a_{ni}| \to 0$ as $n \to \infty$.*

Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - \mathbf{E}X_{ni}) \to 0$$

in L_r, and hence in probability as $n \to \infty$.

THEOREM B. *Suppose that* $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *is an array of rowwise NA h-integrable with exponent* $1 \leqslant r < 2$ *random variables,* $k_n \to \infty$, $h(n) \uparrow \infty$, *and* $h(n)/k_n \to 0$ *as* $n \to \infty$. *Then*

$$\frac{\sum_{i=u_n}^{v_n} (X_{ni} - \mathbf{E}X_{ni})}{k_n^{1/r}} \to 0$$

in L_r, *and hence in probability as* $n \to \infty$.

Inspired by the concept of h-integrability with exponent r, Wang and Hu [35] introduced a new and weaker concept of uniform integrability as follows.

DEFINITION 3.3. *Let* $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *be an array of random variables and* $r > 0$. *The array* $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *is said to be* residually h-integrable (R-h-integrable) *with exponent r if*

$$\sup_{n \geqslant 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbf{E}|X_{ni}|^r < \infty$$

and

$$\lim_{n \to \infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbf{E}\big(|X_{ni}| - h^{1/r}(n)\big)^r \mathbf{I}(|X_{ni}|^r > h(n)) = 0.$$

Under the assumption of R-h-integrability with exponent r, Wang and Hu [35] established some weak laws of large numbers for arrays of dependent random variables. Note that

$$\big(|X_{ni}| - h^{1/r}(n)\big)^r \mathbf{I}(|X_{ni}|^r > h(n)) \leqslant |X_{ni}|^r \mathbf{I}(|X_{ni}|^r > h(n)),$$

and hence the concept of R-h-integrability with exponent r is weaker than h-integrability with exponent r.

As in the case of h-integrability with exponent r, the main idea underlying the notion of R-h-integrability with exponent r deals with the usual normed sums of random variables. We now introduce a new and weaker concept of integrability which deals with weighted sums of random variables.

DEFINITION 3.4. *Let* $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *be an array of random variables, and let* $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *be an array of constants. Let* $r > 0$. *The array* $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *is said to be* R-h-integrable with exponent r concerning the array $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *if*

$$\sup_{n \geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r \mathbf{E}|X_{ni}|^r < \infty$$

and

$$\lim_{n \to \infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r \mathbf{E}\big(|X_{ni}| - h^{1/r}(n)\big)^r \mathbf{I}(|X_{ni}|^r > h(n)) = 0.$$

For $r = 1$ the notion of R-h-integrability with exponent r concerning the array of constants $\{a_{ni}\}$ reduces to the so-called R-h-integrability concerning the array of constants $\{a_{ni}\}$. For more details about the L_r-convergence for weighted sums of random variables based on R-h-integrability, the reader can refer to [48], [25], and so on.

The main purpose of this section is to generalize and improve the results of Theorem A and Theorem B for arrays of rowwise NA random variables to the case of arrays of rowwise m-NOD random variables under some weaker conditions. In addition, we will study the L_r-convergence and weak law of large numbers for a class of random variables under the condition of h-integrability with exponent $1 \leqslant r < 2$, which generalize the corresponding ones of [44] and [30], and improve the corresponding one of [6]. The key techniques used here are the Marcinkiewicz–Zygmund-type inequality and the truncated method.

Our main results on L_r-convergence and weak law of large numbers for arrays of rowwise m-NOD are as follows. The first one is based on the condition of R-h-integrability with exponent $1 \leqslant r < 2$ concerning the array of constants $\{a_{ni}\}$.

THEOREM 3.1. *Let $1 \leqslant r < 2$. Let $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of rowwise m-NOD random variables, and let $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of constants. Assume that the following conditions hold*:

(i) *$\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ is R-h-integrable with exponent r concerning the array of constants $\{a_{ni}\}$, i.e.*,

$$\sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r \mathbf{E}|X_{ni}|^r < \infty$$

and

$$\lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r \mathbf{E}\big(|X_{ni}| - h^{1/r}(n)\big)^r \mathbf{I}(|X_{ni}|^r > h(n)) = 0;$$

(ii) *$h(n)\sup_{u_n\leqslant i\leqslant v_n} |a_{ni}|^r \to 0$ as $n \to \infty$.*

Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - \mathbf{E}X_{ni}) \to 0$$

in L_r, and hence in probability as $n \to \infty$.

Proof. Since $a_{ni} = a_{ni}^+ - a_{ni}^-$, without loss of generality, we assume that $a_{ni} \geqslant 0$. For fixed $n \geqslant 1$, denote for $u_n \leqslant i \leqslant v_n$ that

$$\begin{aligned} Y_{ni} &= -h^{1/r}(n)\mathbf{I}\big(X_{ni} < -h^{1/r}(n)\big) + X_{ni}\mathbf{I}\big(|X_{ni}| \leqslant h^{1/r}(n)\big) \\ &\quad + h^{1/r}(n)\mathbf{I}\big(X_{ni} > h^{1/r}(n)\big), \\ Z_{ni} &= X_{ni} - Y_{ni} = \big(X_{ni} + h^{1/r}(n)\big)\mathbf{I}\big(X_{ni} < -h^{1/r}(n)\big) \\ &\quad + \big(X_{ni} - h^{1/r}(n)\big)\mathbf{I}\big(X_{ni} > h^{1/r}(n)\big), \\ S_n &= \sum_{i=u_n}^{v_n} a_{ni}(Y_{ni} - \mathbf{E}Y_{ni}), \quad T_n = \sum_{i=u_n}^{v_n} a_{ni}(Z_{ni} - \mathbf{E}Z_{ni}). \end{aligned}$$

Note that

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - \mathbf{E}X_{ni}) = S_n + T_n, \qquad n \geqslant 1;$$

we have by the C_r-inequality that

$$\mathbf{E}\left|\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - \mathbf{E}X_{ni})\right|^r \leqslant C\mathbf{E}|S_n|^r + C\mathbf{E}|T_n|^r.$$

To prove $\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - \mathbf{E}X_{ni}) \to 0$ in L_r, we only need to show $\mathbf{E}|S_n|^r \to 0$ and $\mathbf{E}|T_n|^r \to 0$ as $n \to \infty$, where $1 \leqslant r < 2$.

First, we will show that $\mathbf{E}|S_n|^r \to 0$ as $n \to \infty$. Since $1 \leqslant r < 2$, it suffices to show $\mathbf{E}S_n^2 \to 0$ as $n \to \infty$.

For fixed $n \geqslant 1$, it follows by Lemma 2.1 that $\{a_{ni}(Y_{ni} - \mathbf{E}Y_{ni}),\, u_n \leqslant i \leqslant v_n\}$ are m-NOD random variables. Since $|Y_{ni}| = \min\{|X_{ni}|, h^{1/r}(n)\}$, it follows by Theorem 2.1 and Remark 2.1 that

$$\begin{aligned}\mathbf{E}S_n^2 &= \mathbf{E}\left|\sum_{i=u_n}^{v_n} a_{ni}(Y_{ni} - \mathbf{E}Y_{ni})\right|^2 \leqslant C\sum_{i=u_n}^{v_n} a_{ni}^2\mathbf{E}Y_{ni}^2\\ &\leqslant Ch^{(2-r)/r}(n)\sup_{n\geqslant 1}|a_{ni}|^{2-r}\sum_{i=u_n}^{v_n}|a_{ni}|^r\mathbf{E}|Y_{ni}|^r\\ &\leqslant C\Big[h(n)\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\Big]^{(2-r)/r}\sum_{i=u_n}^{v_n}|a_{ni}|^r\mathbf{E}|X_{ni}|^r \to 0 \text{ as } n\to\infty,\end{aligned}$$

which implies that $\mathbf{E}S_n^2 \to 0$ as $n \to \infty$, and thus $\mathbf{E}|S_n|^r \to 0$ as $n \to \infty$.

Further, we will show that $\mathbf{E}|T_n|^r \to 0$ as $n \to \infty$.

For fixed $n \geqslant 1$, it follows by Lemma 2.1 again that $\{a_{ni}(Z_{ni} - \mathbf{E}Z_{ni}),\, u_n \leqslant i \leqslant v_n\}$ are m-NOD random variables. Note that

$$|Z_{ni}| = \big(|X_{ni}| - h^{1/r}(n)\big)\mathbf{I}\big(|X_{ni}| > h^{1/r}(n)\big);$$

we have by Theorem 2.1 and Remark 2.1 again that

$$\begin{aligned}\mathbf{E}|T_n|^r &= \mathbf{E}\left|\sum_{i=u_n}^{v_n} a_{ni}(Z_{ni} - \mathbf{E}Z_{ni})\right|^r \leqslant C\sum_{i=u_n}^{v_n}|a_{ni}|^r\mathbf{E}|Z_{ni}|^r\\ &= C\sum_{i=u_n}^{v_n}|a_{ni}|^r\mathbf{E}\big(|X_{ni}| - h^{1/r}(n)\big)^r\mathbf{I}\big(|X_{ni}|^r > h(n)\big) \to 0 \text{ as } n\to\infty,\end{aligned}$$

which implies that $\mathbf{E}|T_n|^r \to 0$ as $n \to \infty$. This completes the proof of the theorem.

If we take $a_{ni} = k_n^{-1/r}$ for $u_n \leqslant i \leqslant v_n$ and $n \geqslant 1$, then we can get the following result on L_r-convergence and weak law of large numbers for arrays of rowwise m-NOD R-h-integrable with exponent $1 \leqslant r < 2$ random variables.

COROLLARY 3.1. *Let $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of rowwise m-NOD R-h-integrable random variables with exponent $1 \leqslant r < 2$, $k_n \to \infty$, $h(n) \uparrow \infty$, and $h(n)/k_n \to 0$ as $n \to \infty$. Then*

$$\frac{1}{k_n^{1/r}}\sum_{i=u_n}^{v_n}(X_{ni} - \mathbf{E}X_{ni}) \to 0$$

in L_r, and hence in probability as $n \to \infty$.

Remark 3.1. Note that

$$\big(|X_{ni}| - h^{1/r}(n)\big)^r \mathbf{I}(|X_{ni}|^r > h(n)) \leqslant |X_{ni}|^r \mathbf{I}(|X_{ni}|^r > h(n)),$$

and $h(n)\sup_{u_n \leqslant i \leqslant v_n} |a_{ni}| \to 0$ implies $h(n)\sup_{u_n \leqslant i \leqslant v_n} |a_{ni}|^r \to 0$ (here, $1 \leqslant r < 2$ and $h(n) \uparrow \infty$ as $n \to \infty$), which imply that the conditions of Theorem 3.1 are weaker than those of Theorem A. Hence, the result of Theorem 3.1 generalizes and improves the corresponding one of Theorem A.

Remark 3.2. Since the concept of R-h-integrability with exponent r is weaker than h-integrability with exponent r, and since m-NOD is weaker than NA, the result of Corollary 3.1 generalizes and improves that of Theorem B.

Further, we will establish the L_r-convergence and weak law of large numbers for a class of random variables satisfying the Marcinkiewicz–Zygmund inequality with exponent 2, which includes m-NOD as a special case. The main ideas are inspired by [6] and [30].

We say that a sequence $\{X_n, n \geqslant 1\}$ of random variables satisfies the Marcinkiewicz–Zygmund inequality with exponent 2 if, for all $n \geqslant 1$,

$$\mathbf{E}\bigg|\sum_{i=1}^{n} X_i\bigg|^2 \leqslant C\sum_{i=1}^{n} \mathbf{E}|X_i|^2,$$

where C is a positive constant not depending on n.

An array $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ of random variables is said to satisfy the Marcinkiewicz–Zygmund inequality with exponent 2 if, for all $n \geqslant 1$,

$$\mathbf{E}\bigg|\sum_{i=u_n}^{v_n} X_{ni}\bigg|^2 \leqslant C\sum_{i=u_n}^{v_n} \mathbf{E}|X_{ni}|^2,$$

where C is a positive constant not depending on n.

Remark 3.3. There are many sequences of mean zero random variables satisfying the Marcinkiewicz–Zygmund inequality with exponent 2; we mention an independent sequence, a martingale difference sequence, a φ-mixing sequence with the mixing coefficients satisfying certain conditions (see [36]), a ρ-mixing sequence with the mixing coefficients satisfying certain conditions (see [20]), a $\widetilde{\rho}$-mixing sequence (see [32]), an NA sequence (see [21]), an NOD sequence (see [2]), an END sequence (see [22]), an NSD sequence (see [11] or [39]), an AANA sequence with the mixing coefficients satisfying certain conditions (see [49]), a ρ^--mixing sequence with the mixing coefficients satisfying certain conditions (see [34]), a pairwise negatively quadrant dependent (PNQD) sequence (see [17]), an m-NOD sequence (see our Theorem 2.1), a linearly negative quadrant dependent (LNQD) sequence [51], and so on.

Our main result on L_r-convergence and weak law of large numbers for a class of random variables satisfying the Marcinkiewicz–Zygmund inequality with exponent 2 is as follows.

THEOREM 3.2. *Let* $1 \leqslant r < 2$. *Let* $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ *be an array of random variables, and let* $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ *be an array of constants. Assume that the following conditions hold*:

(i) $\sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r \mathbf{E}|X_{ni}|^r < \infty$;

(ii) *for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r \mathbf{E}|X_{ni}|^r \mathbf{I}(|X_{ni}|^r > \varepsilon) = 0;$$

(iii) *for any* $t > 0$, *the array* $\{Y_{ni} - \mathbf{E}Y_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *satisfies the Marcinkiewicz–Zygmund inequality with exponent* 2, *where*

$$Y_{ni} = a_{ni}X_{ni}\mathbf{I}\big(|a_{ni}X_{ni}| \leqslant t^{1/r}\big)$$

or

$$Y_{ni} = -t^{1/r}\mathbf{I}\big(a_{ni}X_{ni} < -t^{1/r}\big) + a_{ni}X_{ni}\mathbf{I}\big(|a_{ni}X_{ni}| \leqslant t^{1/r}\big) + t^{1/r}\mathbf{I}\big(a_{ni}X_{ni} > t^{1/r}\big).$$

Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - \mathbf{E}X_{ni}) \to 0$$

in L_r, *and hence in probability as* $n \to \infty$.

Proof. The proof is similar to that of [30]. The details are omitted.

With Theorem 3.2 in hand, the following corollary can be proved as Corollary 2.1 of [30].

COROLLARY 3.2. *Let* $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *be an array of constants satisfying* $k_n \doteq 1/\sup_{u_n\leqslant i\leqslant v_n} |a_{ni}|^r \to \infty$, $0 < h(n) \uparrow \infty$, *and* $h(n)/k_n \to 0$ *as* $n \to \infty$. *Let* $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *be an array of* h*-integrable random variables with exponent* $1 \leqslant r < 2$. *Assume further that condition* (iii) *in Theorem* 3.2 *holds. Then*

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - \mathbf{E}X_{ni}) \to 0$$

in L_r, *and hence in probability as* $n \to \infty$.

Taking $a_{ni} = k_n^{-1/r}$ for $u_n \leqslant i \leqslant v_n$ and $n \geqslant 1$ in Corollary 3.2, we readily get the following result.

COROLLARY 3.3. *Let* $\{X_{ni}, u_n \leqslant i \leqslant v_n, n\geqslant 1\}$ *be an array of* h*-integrable with exponent* $1 \leqslant r < 2$ *random variables*, $k_n \to \infty$, $0 < h(n) \uparrow \infty$, *and* $h(n)/k_n \to 0$ *as* $n \to \infty$. *Assume further that the condition* (iii) *in Theorem* 3.2 *holds*, *where* $a_{ni} = k_n^{-1/r}$ *for* $u_n \leqslant i \leqslant v_n$ *and* $n \geqslant 1$. *Then*

$$\frac{\sum_{i=u_n}^{v_n}(X_{ni} - \mathbf{E}X_{ni})}{k_n^{1/r}} \to 0$$

in L_r, *and hence in probability as* $n \to \infty$.

Remark 3.4. We have pointed out that a PNQD sequence satisfies the Marcinkiewicz–Zygmund inequality with exponent 2 in Remark 3.3. Hence, the results of Theorem 3.2 and Corollaries 3.2 and 3.3 in the present paper extend the corresponding ones of Theorem 2.1 and Corollaries 2.1 and 2.2 for PNQD random variables in [30], respectively. In addition, note that LNQD implies PNQD (see [30]), and hence our results of Corollaries 3.2 and 3.3 generalize the corresponding ones of Theorem 3.1 and Corollary 3.1 for LNQD random variables in [44], respectively.

Remark 3.5. Under the conditions of Corollary 3.2, Chen, Cabrera, and Volodin [6] established the L_1-convergence and the weak law of large numbers for arrays of rowwise h-integrable with exponent $r = 1$ random variables satisfying the Marcinkiewicz–Zygmund inequality with exponent 2. Here, we put forward the L_r-convergence and the weak law of large numbers for arrays of rowwise h-integrable with exponent $1 \leqslant r < 2$ random variables satisfying the Marcinkiewicz–Zygmund inequality with exponent 2. In addition, the condition "$k_n \doteq 1/\sup_{u_n \leqslant i \leqslant v_n} |a_{ni}|^r \to \infty$, $0 < h(n) \uparrow \infty$ and $h(n)/k_n \to 0$ as $n \to \infty$" in Corollary 3.2 in the paper is weaker than the condition "$k_n \doteq 1/\sup_{u_n \leqslant i \leqslant v_n} |a_{ni}| \to \infty$, $0 < h(n) \uparrow \infty$ and $h(n)/k_n \to 0$ as $n \to \infty$" in Theorem 1 in [6]. Hence, our results of Theorem 3.2 and Corollary 3.2 generalize and improve the corresponding Theorem 1 of [6].

3.2. Strong convergence. In section 3.1, we studied the L_r-convergence and the weak law of large numbers for arrays of rowwise m-NOD random variables under some uniformly integrable conditions. In order to establish the strong version of Theorem 3.1, we introduce the concept of strongly residual h-integrability with exponent r concerning the array of constants $\{a_{ni}\}$ as follows.

DEFINITION 3.5. *Let $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of random variables, and let $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of constants. Let $r > 0$. The array $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ is said to be* strongly residually h-integrable (SR-h-integrable) *with exponent r concerning the array $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ if*

$$\sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r \mathbf{E}|X_{ni}|^r < \infty$$

and

$$\sum_{n=1}^{\infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r \mathbf{E}\big(|X_{ni}| - h^{1/r}(n)\big)^r \mathbf{I}\big(|X_{ni}|^r > h(n)\big) < \infty.$$

For $r = 1$ the preceding definition reduces to the concept of SR-h-integrability concerning the array $\{a_{ni}, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$, which was introduced by Cabrera, Rosalsky, and Volodin [18].

The main idea of the notion of SR-h-integrability with exponent r concerning the array $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ is to deal with weighted sums of random variables. We introduce a new concept of integrability, which deals with usual normed sums of random variables as follows.

DEFINITION 3.6. *Let $\{X_{ni}, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ be an array of random variables and $r > 0$. The array $\{X_{ni}, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ is said to be* SR-h-integrable with exponent r *if*

$$\sup_{n\geqslant 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbf{E}|X_{ni}|^r < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbf{E}\big(|X_{ni}| - h^{1/r}(n)\big)^r \mathbf{I}\big(|X_{ni}|^r > h(n)\big) < \infty.$$

Remark 3.6. It is easily seen that the concept of SR-h-integrability with exponent r is stronger than the concept of R-h-integrability with exponent r.

Our main result on strong convergence for weighted sums of arrays of rowwise m-NOD random variables under some uniformly integrable conditions is as follows.

THEOREM 3.3. *Let* $1 \leqslant r < 2$. *Let* $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ *be an array of rowwise m-NOD random variables, and let* $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ *be an array of constants. Assume that the following conditions hold*:

(i) $\{X_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$ *is SR-h-integrable with exponent r concerning the array* $\{a_{ni},\, u_n \leqslant i \leqslant v_n,\, n \geqslant 1\}$;

(ii)

$$\sum_{n=1}^{\infty}\Big(h(n)\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\Big)^{(2-r)/r} < \infty.$$

Then $\sum_{i=u_n}^{v_n} a_{ni}(X_{ni}-\mathbf{E}X_{ni}) \to 0$ *a.s. as* $n\to\infty$.

Proof. We use the same notation as in Theorem 3.1. In order to prove that $\sum_{i=u_n}^{v_n} a_{ni}(X_{ni}-\mathbf{E}X_{ni}) \to 0$ a.s. as $n\to\infty$ it suffices to show that

$$S_n \doteq \sum_{i=u_n}^{v_n} a_{ni}(Y_{ni}-\mathbf{E}Y_{ni}) \to 0 \quad \text{a.s. as } n\to\infty \tag{3.1}$$

and

$$T_n \doteq \sum_{i=u_n}^{v_n} a_{ni}(Z_{ni}-\mathbf{E}Z_{ni}) \to 0 \quad \text{a.s. as } n\to\infty. \tag{3.2}$$

First, we will prove (3.1). Note that $|Y_{ni}| = \min\{|X_{ni}|, h^{1/r}(n)\}$; using Markov's inequality, Theorem 2.1 (or Remark 2.1), Jensen's inequality, and conditions (i), (ii) we have for any $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbf{P}(|S_n|>\varepsilon) &\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\mathbf{E}\Big|\sum_{i=u_n}^{v_n} a_{ni}(Y_{ni}-\mathbf{E}Y_{ni})\Big|^2 \leqslant C\sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n} a_{ni}^2\mathbf{E}Y_{ni}^2\\ &\leqslant C\sum_{n=1}^{\infty} h^{(2-r)/r}(n)\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^{2-r}\sum_{i=u_n}^{v_n}|a_{ni}|^r\mathbf{E}|Y_{ni}|^r\\ &\leqslant C\sum_{n=1}^{\infty}\Big[h(n)\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\Big]^{(2-r)/r}\left(\sup_{n\geqslant 1}\sum_{i=u_n}^{v_n}|a_{ni}|^r\mathbf{E}|X_{ni}|^r\right) < \infty,\end{aligned}$$

which implies (3.1) by the Borel–Cantelli lemma.

In the following we will prove (3.2). Note that

$$|Z_{ni}| = (|X_{ni}| - h^{1/r}(n))\mathbf{I}\big(|X_{ni}|^r > h(n)\big);$$

it follows by Markov's inequality, Theorem 2.1 (or Remark 2.1), Jensen's inequality, and condition (i) that for any $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbf{P}(|T_n|>\varepsilon) &\leqslant \frac{1}{\varepsilon^r}\sum_{n=1}^{\infty}\mathbf{E}\Big|\sum_{i=u_n}^{v_n} a_{ni}(Z_{ni}-\mathbf{E}Z_{ni})\Big|^r \leqslant C\sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n}|a_{ni}|^r\mathbf{E}|Z_{ni}|^r\\ &= C\sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n}|a_{ni}|^r\mathbf{E}\big(|X_{ni}|-h^{1/r}(n)\big)^r\mathbf{I}\big(|X_{ni}|^r>h(n)\big) < \infty,\end{aligned}$$

which implies (3.2) by the Borel–Cantelli lemma. This completes the proof of the theorem.

If we take $a_{ni} = k_n^{-1/r}$ for $u_n \leqslant i \leqslant v_n$ and $n \geqslant 1$ in Theorem 3.3, then we can get the following result on strong convergence for arrays of rowwise m-NOD SR-h-integrable with exponent $1 \leqslant r < 2$ random variables.

COROLLARY 3.4. *Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of rowwise m-NOD SR-h-integrable random variables with exponent $1 \leqslant r < 2$, $k_n \to \infty$, $h(n) \uparrow \infty$, and*

$$\sum_{n=1}^{\infty} \left(\frac{h(n)}{k_n} \right)^{(2-r)/r} < \infty.$$

Then

$$\frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (X_{ni} - \mathbf{E}X_{ni}) \to 0 \quad a.s.$$

as $n \to \infty$.

4. On the asymptotic approximation of inverse moment for nonnegative *m*-NOD random variables. As an application of the moment inequalities for m-NOD random variables, section 3 deals with the L_r-convergence and strong convergence for m-NOD random variables under some uniformly integrable conditions. As another application of the moment inequalities for m-NOD random variables, in this section we will study the asymptotic approximation of inverse moments for nonnegative m-NOD random variables with finite first moments.

Let $Z_1, Z_2, \ldots$ be a sequence of nonnegative random variables with finite second moments. Denote

$$X_n = \frac{1}{B_n} \sum_{i=1}^{n} Z_i, \qquad B_n^2 = \sum_{i=1}^{n} D(Z_i). \tag{4.1}$$

Under some suitable conditions, the inverse moment can be approximated by the inverse of the moment in the following way:

$$\mathbf{E}(a + X_n)^{-\alpha} \sim (a + \mathbf{E}X_n)^{-\alpha}, \tag{4.2}$$

where $a > 0$ and $\alpha > 0$ are arbitrary real numbers. Here and in what follows, for two positive sequences $\{c_n, n \geqslant 1\}$ and $\{d_n, n \geqslant 1\}$, we write $c_n \sim d_n$ and $c_n = o(d_n)$ if $\lim_{n\to\infty} c_n d_n^{-1} = 1$, $\lim_{n\to\infty} c_n d_n^{-1} = 0$. The left-hand side of (4.2) is the inverse moment, and the right-hand side is the inverse of the moment. Usually, the inverse of the moment is much easier to compute than the inverse moment. So in many practical applications, such as evaluating risks of estimators and powers of tests, reliability, life testing, insurance, and financial mathematics, complex systems, and so on, we often take the inverse of the moment instead of the inverse moment. Up to now, many authors studied the asymptotic approximation of inverse moment and found many interesting results. For the details about the inverse moment, the reader can refer to [5], [7], [8], [15], [43], [37], [26], [24], [47] [9], [27], etc.

For $n \geqslant 1$, denote

$$\widetilde{X}_n = \sum_{i=1}^{n} Z_i, \qquad \widetilde{\mu}_n = \mathbf{E}\widetilde{X}_n, \tag{4.3}$$

and

$$\mu_{n,s} = \sum_{i=1}^{n} \mathbf{E} Z_i \mathbf{I}(Z_i \leqslant \mu_n^s/\sqrt{n}) \quad \text{for some } 0 < s < 1.$$

Based on the above notation, Shi, Wu, and Liu [26] obtained the following Theorem C and Theorem D.

THEOREM C. *Let $\{Z_n, n \geqslant 1\}$ be a sequence of independent, nonnegative, and nondegenerated random variables. Assume that the following conditions hold*:

(H_1) *$\widetilde{\mu}_n \to \infty$ as $n \to \infty$;*

(H_2) *$\widetilde{\mu}_n \sim \mu_{n,s}$ for some $0 < s < 1$.*

Then

$$\mathbf{E}(a + \widetilde{X}_n)^{-\alpha} \sim (a + \mathbf{E}\widetilde{X}_n)^{-\alpha} \tag{4.4}$$

holds for all real constants $a > 0$ and $\alpha > 0$.

THEOREM D. *Let the conditions of Theorem* C *hold. In addition, suppose that there exists a function $f(x)$, $x \geqslant 0$, satisfying the following conditions*:

(H_3) *There exists a $c_1 > 0$ such that $f(x) > c_1$ for $x \geqslant 0$;*

(H_4) *there exist $k > 0$ and $c_2 > 0$ such that $f(x)/x^k \to c_2$ as $x \to \infty$;*

(H_5) *$1/f(x)$ is a convex function for $x \geqslant 0$.*

Then

$$\mathbf{E}[f(\widetilde{X}_n)]^{-1} \sim [f(\mathbf{E}\widetilde{X}_n)]^{-1}. \tag{4.5}$$

Denote

$$\widetilde{\mu}_{n,s} = \sum_{i=1}^{n} \mathbf{E} Z_i \mathbf{I}(Z_i \leqslant \mu_n^s) \quad \text{for some } 0 < s < 1.$$

Consider the following assumption:

(H_6) $\mu_n \sim \widetilde{\mu}_{n,s}$ for some $0 < s < 1$.

Yang, Hu, and Wang [47] pointed out that condition (H_6) is weaker than (H_2) and extended Theorem C for independent random variables to the case of nonnegative random variables under conditions (H_1) and (H_6).

Recently, Shen [24] generalized the result of Theorem C to a general case and obtained the following result.

THEOREM E. *Let $\{Z_n, n \geqslant 1\}$ be a sequence of nonnegative random variables with $\mathbf{E}Z_n < \infty$ for all $n \geqslant 1$ and $0 < s < 1$. Let $\{M_n, n \geqslant 1\}$ and $\{a_n, n \geqslant 1\}$ be sequences of positive constants such that $a_n \geqslant C$ for all n sufficiently large, where C is a positive constant. Denote $X_n = M_n^{-1} \sum_{k=1}^n Z_k$ and $\mu_n = \mathbf{E}X_n$ and $D_n = \eta M_n \mu_n^s / a_n$, where η is a positive constant. Suppose that the following conditions hold*:

(i) *For any $p > 2$, there exist positive constants η and C (depending only on p) such that*

$$\mathbf{E}\left|\sum_{k=1}^{n}(Z'_{nk} - \mathbf{E}Z'_{nk})\right|^p \leqslant C\left[\sum_{k=1}^{n} \mathbf{E}|Z'_{nk} - \mathbf{E}Z'_{nk}|^p + \left(\sum_{k=1}^{n} \mathrm{Var}(Z'_{nk})\right)^{p/2}\right],$$

where $Z'_{nk} = Z_k \mathbf{I}(Z_k \leqslant D_n) + D_n \mathbf{I}(Z_k > D_n)$, or $Z'_{nk} = Z_k \mathbf{I}(Z_k \leqslant D_n)$;

(ii) *$\mu_n \to \infty$ as $n \to \infty$;*

(iii)

$$\frac{\sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n)}{\sum_{k=1}^{n} \mathbf{E} Z_k} \to 0 \quad \textit{as } n \to \infty,$$

where $\eta > 0$ *is the same as that in* (i).

Then (4.2) *holds for all real constants* $a > 0$ *and* $\alpha > 0$.

Inspired by the results alluded to above, we will establish the asymptotic approximation of inverse moment as follows.

THEOREM 4.1. *Let the conditions of Theorem* E *and* (H_3)–(H_5) *hold. In addition, assume that there exists a positive constant* γ *such that* $f(x)$ *is a nondecreasing function for* $x \geqslant \gamma$. *Then*

$$\mathbf{E}[f(X_n)]^{-1} \sim [f(\mathbf{E}X_n)]^{-1}. \tag{4.6}$$

Proof. Applying Jensen's inequality to the convex function $1/f(x)$, we have

$$\mathbf{E}[f(X_n)]^{-1} \geqslant [f(\mathbf{E}X_n)]^{-1},$$

which implies that

$$\liminf_{n\to\infty} f(\mathbf{E}X_n)\mathbf{E}[f(X_n)]^{-1} \geqslant 1. \tag{4.7}$$

To prove (4.6), we only need to show that

$$\limsup_{n\to\infty} f(\mathbf{E}X_n)\mathbf{E}[f(X_n)]^{-1} \leqslant 1. \tag{4.8}$$

For any $0 < \delta < 1$, let

$$U_n = M_n^{-1} \sum_{k=1}^{n} \big[Z_k \mathbf{I}(Z_k \leqslant D_n) + \eta M_n \mu_n^s / a_n \mathbf{I}(Z_k > D_n)\big] \doteq M_n^{-1} \sum_{k=1}^{n} Z'_{nk}$$

and

$$\begin{aligned} \mathbf{E}[f(X_n)]^{-1} &= \mathbf{E}[f(X_n)]^{-1}\mathbf{I}(U_n \geqslant \mu_n - \delta\mu_n) + \mathbf{E}[f(X_n)]^{-1}\mathbf{I}(U_n < \mu_n - \delta\mu_n) \\ &\doteq Q_1 + Q_2. \end{aligned} \tag{4.9}$$

Noting that $X_n \geqslant U_n$ and $f(x)$ is a nondecreasing function for $x \geqslant \gamma$, we have by (H_4)

$$\begin{aligned} \limsup_{n\to\infty} f(\mathbf{E}X_n)Q_1 &\leqslant \limsup_{n\to\infty} f(\mu_n)\mathbf{E}[f(U_n)]^{-1}\mathbf{I}(U_n > \mu_n - \delta\mu_n) \\ &\leqslant \limsup_{n\to\infty} \left[\frac{f(\mu_n)}{\mu_n^k} \cdot \frac{\mu_n^k}{(\mu_n - \delta\mu_n)^k} \cdot \frac{(\mu_n - \delta\mu_n)^k}{f(\mu_n - \delta\mu_n)}\right] \\ &= (1-\delta)^{-k} \to 1 \quad \text{as } \delta \downarrow 0. \end{aligned} \tag{4.10}$$

In the following, we will prove that

$$\lim_{n\to\infty} f(\mathbf{E}X_n)Q_2 = 0. \tag{4.11}$$

For $0 < \delta < 1$ given above, it follows by (iii) in Theorem E that there exists a positive integer $n(\delta) > 0$ such that

$$\sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n) \leqslant \frac{\delta}{4} \sum_{k=1}^{n} \mathbf{E} Z_k, \qquad n \geqslant n(\delta), \tag{4.12}$$

and hence for $n \geqslant n(\delta)$,

$$\begin{aligned} |\mu_n - \mathbf{E} U_n| &= \left| M_n^{-1} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n) - M_n^{-1} \sum_{k=1}^{n} D_n \mathbf{E} \mathbf{I}(Z_k > D_n) \right| \\ &\leqslant M_n^{-1} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n) + M_n^{-1} \sum_{k=1}^{n} D_n \mathbf{E} \mathbf{I}(Z_k > D_n) \\ &\leqslant M_n^{-1} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n) + M_n^{-1} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n) \\ &= 2 M_n^{-1} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n) \leqslant \frac{\delta \mu_n}{2}. \end{aligned} \tag{4.13}$$

By condition (H_3), (4.13), Markov's inequality, condition (i) in Theorem E, and the C_r-inequality, we have for any $p > 2$ and all $n \geqslant n(\delta)$,

$$\begin{aligned} Q_2 &\leqslant C \mathbf{P}\left(|U_n - \mathbf{E} U_n| > \frac{\delta \mu_n}{2} \right) \leqslant C \mu_n^{-p} M_n^{-p} \mathbf{E} \left| \sum_{k=1}^{n} (Z'_{nk} - \mathbf{E} Z'_{nk}) \right|^p \\ &\leqslant C \mu_n^{-p} \left(M_n^{-2} \sum_{k=1}^{n} \mathbf{E} Z_k^2 \mathbf{I}(Z_k \leqslant D_n) + M_n^{-2} \sum_{k=1}^{n} D_n^2 \mathbf{E} \mathbf{I}(Z_k > D_n) \right)^{p/2} \\ &\quad + C \mu_n^{-p} \left[M_n^{-p} \sum_{k=1}^{n} \mathbf{E} Z_k^p \mathbf{I}(Z_k \leqslant D_n) + M_n^{-p} \sum_{k=1}^{n} D_n^p \mathbf{E} \mathbf{I}(Z_k > D_n) \right] \\ &\leqslant C \mu_n^{-p} \left(M_n^{-1} \frac{\mu_n^s}{a_n} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k \leqslant D_n) + M_n^{-1} \frac{\mu_n^s}{a_n} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n) \right)^{p/2} \\ &\quad + C \mu_n^{-p} M_n^{-1} \frac{\mu_n^{s(p-1)}}{a_n^{p-1}} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k \leqslant D_n) \\ &\quad + C \mu_n^{-p} M_n^{-1} \frac{\mu_n^{s(p-1)}}{a_n^{p-1}} \sum_{k=1}^{n} \mathbf{E} Z_k \mathbf{I}(Z_k > D_n) \\ &= C \left[\frac{\mu_n^{-(1-s)p/2}}{a_n^{p/2}} + \frac{\mu_n^{-(1-s)(p-1)}}{a_n^{p-1}} \right] \leqslant C \mu_n^{-(1-s)p/2} + C \mu_n^{-(1-s)(p-1)}. \end{aligned} \tag{4.14}$$

Note that $p > 2$, and thus $p - 1 > p/2$. Taking $p > \max\{2, 2k/(1-s)\}$, we have by (4.14) that $Q_2 = o(\mu_n^{-k})$ which, together with (H_4), yields (4.11). Now (4.8) readily follows from (4.9)–(4.11).

A similar argument establishes (4.6) for

$$U_n = M_n^{-1} \sum_{k=1}^{n} Z_k \mathbf{I}(Z_k \leqslant D_n) \doteq M_n^{-1} \sum_{k=1}^{n} Z'_{nk}.$$

This completes the proof of the theorem.

Remark 4.1. Since the Rosenthal-type inequality (i.e., condition (i) in Theorem E) is satisfied for m-NOD random variables, the result of Theorem 4.1 holds for nonnegative m-NOD random variables and other random variables, such as ρ-mixing random variables, φ-mixing random variables, $\widetilde{\rho}$-mixing random variables, NA random variables, NSD random variables, NOD random variables, END random variables, AANA random variables, ρ^--mixing random variables, and so on.

Remark 4.2. We take $f(x) = (a+x)^\alpha$, $x \geqslant 0$, $a > 0$, and $\alpha > 0$. It is easy to check that $f(x)$ is a nondecreasing function for $x \geqslant 0$ and that conditions (H_3)–(H_5) hold. Hence, Theorem E can be obtained by Theorem 4.1 easily. That is, Theorem E is a special case of Theorem 4.1. In addition, if we take $M_n \equiv 1$ and $a_n = \eta\sqrt{n}$, then we can get Theorem D immediately; if we take $M_n \equiv 1$ and $a_n = \eta$, then we can readily get Theorem 2.3 of [47]; if we take $a_n = u_n^s$, then we can get Theorem 2.2 of [9] immediately. Therefore, our result of Theorem 4.1 generalizes the corresponding one of [26], [24], [47], and [9].

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