Eigenvalue Methods in Erdős-Ko-Rado Combinatorics

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Intersecting Families of Sets

Definition

An **intersecting k-set family on an n-set** is

1. a collection of subsets from $\{1, 2, \ldots, n\}$,
2. each of size k, with
3. any two subsets from the family have at least one element in common.

A family is **t-intersecting** if any two sets from the system have at least t elements in common.

Example (A intersecting (7,3)-family)

$$\{\mathbf{1,2,3}\}, \{\mathbf{1,2},4\}, \{\mathbf{1,2},5\}, \{\mathbf{1,2},6\}, \{\mathbf{1,2},7\},$$
$$\{\mathbf{1,3},4\}, \{\mathbf{1,3},5\}, \{\mathbf{1,3},6\}, \{\mathbf{1,3},7\}, \{\mathbf{2,3},4\},$$
$$\{\mathbf{2,3},5\}, \{\mathbf{2,3},6\}, \{\mathbf{2,3},7\}$$

Every set has at least 2 elements from $\{1,2,3\}$ with size $\binom{n-3}{k-3} + \binom{3}{2}\binom{n-3}{k-2}$.

Canonical Intersecting Families

Example (Another intersecting (7,3)-family)

$$\{\mathbf{1},2,3\},\{\mathbf{1},2,4\},\{\mathbf{1},2,5\},\{\mathbf{1},2,6\},\{\mathbf{1},2,7\},$$
$$\{\mathbf{1},3,4\},\{\mathbf{1},3,5\},\{\mathbf{1},3,6\},\{\mathbf{1},3,7\},\{\mathbf{1},4,5\},$$
$$\{\mathbf{1},4,6\},\{\mathbf{1},4,7\},\{\mathbf{1},5,6\},\{\mathbf{1},5,7\},\{\mathbf{1},6,7\}$$

Every set contains 1 size $\binom{n-1}{k-1}$.

The family of all sets that contains a common element has size $\binom{n-1}{k-1}$; is called **canonically** or **trivially intersecting** or a **star** or **point pencil**.

The family of all sets that contains a common t-subset has size $\binom{n-t}{k-t}$; is called **canonically t-intersecting**.

Erdős-Ko-Rado Theorem

Theorem (Erdős-Ko-Rado, EKR)

Let $\mathcal{F}$ be a t-intersecting k-set system on an n-set. If $n > f(k,t)$, then

1. $|\mathcal{F}| \leq \binom{n-t}{k-t}$,
2. *and $\mathcal{F}$ meets this bound if and only if $\mathcal{F}$ is canonically t-intersecting.*

- For $t = 1$ this bound is $f(k,1) = 2k$.
- 1961 - Erdős, Ko and Rado had $f(k,t) \geq t + (k-t)\binom{k}{t}^3$ (but knew it wasn't exact).
- 1978 - Frankl proved $f(k,t) = (t+1)(k-t+1)$ when t is large [F].
- 1984 - Wilson gave an algebraic proof with $f(k,t) = (t+1)(k-t+1)$ for all t [W].
- 1997 - Ahslwede and Khachatrian found the largest t-intersecting family every triple t, k, n [AK].

Quotes about Erdős-Ko-Rado Theorems

> *'While the notion of an intersecting set system is rather dry on the face of things, it applies broadly to many areas of mathematics, as sets are virtually everywhere.*

From Jon Swenson https://jmswen.wordpress.com/2011/06/05/a-probabilistic-proof-of-the-erdos-ko-rado-theorem/

> *Entering the keywords Erdős-Ko-Rado into MathSciNet gives a list of* **80 articles** *presenting Erdős-Ko-Rado results, but we should also count the original article of Erdős et al., which is not included since its title does not contain the words Erdős-Ko-Rado.*

From "Theorems of Erdős-Ko-Rado type in geometrical settings", Maarten de Boeck, Leo Storme, 2013.

My last search on MathSciNet revealed **1,416 articles** Erdős-Ko-Rado results. There are several surveys [DF, E, FT, GM].

EKR for Other Objects

Object	Definition of intersection
k-Sets	a common element
Blocks in a design	a common element
Multi-sets	a common element
Vector spaces over a field	a common 1-D subspace
Lines in a partial geometry	a common point
Different geometries ☕	common subspaces
Integer sequences	same entry in same position
Permutations	both map i to j
Permutations	a common cycle
Permutations in a group ☕	both map i to j
Set Partitions	a common class
Trees on n vertices	a common edge
Cocliques in a graph ☕	a common vertex
Triangulations of a polygon ☠	a common triangle

For any object with a type of intersection, what is the largest intersecting family?

General Framework

- Each **object** is made of k **atoms**.

Object	Atoms
Sets	elements from $\{1, \ldots, n\}$
Blocks of a design	elements from $\{1, \ldots, n\}$
multi-sets	elements from $\{1, \ldots, n\}$
Lines in a partial geometry	points in geometry
Integer sequences	pairs (i, a) (entry a is in position i)
Permutations	pairs (i, j) (the permutation maps i to j)
Permutations	cycles
Set partitions	subsets (cells in the partition)
Trees	$n - 1$ edges
Cocliques in a graph	vertices
Triangulations of a polygon	Triangles in the Triangulations

- Two objects **intersect** if they contain a common atom.

Canonically intersecting family is a collection of all objects with a fixed atom.

EKR-type Questions

1. What is the **size** of the largest intersecting families of objects?
2. **Characterize** all the intersecting families of largest size.

EKR property: a canonically intersecting family is a largest intersecting family.

Strict EKR property: only the canonically intersecting families are the largest intersecting families.

Related Intersection questions:

1. What is the size of the largest family so that any two sets have size **exactly** t.
2. r**-wise intersection**, so what is the largest family in which r of the objects have non-empty intersection.
3. Two families of objects from $[n]$, $\mathcal{A}$ and $\mathcal{B}$ are **cross-intersecting** if for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$ we have $A \cap B \neq \emptyset$. What are the largest cross-intersecting families?
   Results on cross-intersecting families can be used to get results on intersecting families—take $\mathcal{A} = \mathcal{B}$.

Derangement Graphs

My preferred method represent the problem in a **graph** and use algebraic graph theory.

For a set of objects, define the **derangement graph** of the object

- the vertices are the objects,
- two vertices are adjacent if they are **not** intersecting.

- A coclique in the derangement graph is an intersecting family of objects.
  A **coclique/independent set** is a set of vertices in which no two are adjacent. The size of the largest coclique in X is denoted by $\alpha(X)$.

What is the size of the maximum coclique in a derangement graph?

Which cocliques achieve this bound?

- The objects have the EKR property if the canonical intersecting families are maximum cocliques (call these **canonical cocliques**).

Related Problems

Related Problems can be phrased a graph properties of the Derangement graph:

Graph Structure	
Largest coclique	Intersecting set of objects.
Largest bipartite graph	largest collection of objects that can be partition into two intersecting sets.
A proper colouring	smallest partition of the objects into intersecting sets.
Largest K_r-free subgraph	Largest collection of objects so that there are no r mutually disjoint objects.

Examples of Derangement Graphs

Often derangement graphs are very symmetric: Often they are

1. vertex-transitive and have a large automorphism group.
2. belong to an association scheme
3. are Cayley graphs.

This make algebraic methods feasible.
Examples:

Object	Graph	Properties
Sets	Kneser graph	Association Scheme
Blocks in a Design	Block Graph (SRG)	Association Scheme
Vector spaces	q-Kneser graph	Association Scheme
Integer sequences	n-Hamming graph	Association Scheme
Permutations	Derangement graph	Association Scheme and Cayley graph
Triangulations of a polygon	$n-3$ distance graph of the associahedron	vertex-transitive

Derangement Graph

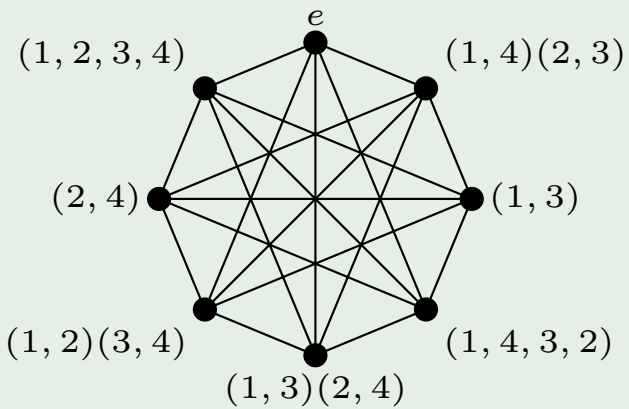
Definition

For any permutation group G we define a **derangement graph**, Γ_G.

- The vertices are the elements of G.
- Vertices $\sigma, \pi \in G$ are adjacent if and only if $\sigma\pi^{-1}$ is a derangement.

Permutations are adjacent if they are **not** intersecting.

Example



The graph $\Gamma_{D(4)}$.

Eigenvalues of Cayley Graphs

1. The derangement graph is a **normal** Cayley graph

$$\Gamma_G = \mathrm{Cay}(G, \mathrm{Der}(G))$$

2. vertices are elements of G; vertices σ, π are adjacent if $\sigma\pi^{-1} \in \mathrm{Der}(G)$
3. The **connection set** of the Cayley graph is set of derangements; so it is closed under conjugation.
4. G is a subgroup of the automorphism group of Γ_G.
5. This graph is **vertex transitive**; the automorphism group acts transitively on the vertices.

This is totally normal



Characterization for Permutations

Theorem

Since Γ_G is a normal Cayley graph, it eigenvalues are

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_i \chi(\sigma)|C_i|$$

where the χ is an irreducible representation of G and the sum is over all conjugacy classes of derangements in G.

The eigenvalues of the graph gives information about its structure.

- The graph is connected if and only if the degree has multiplicity 1.
- Γ_G is connected if and only if the derangements generate the group.
  When this happens is open in general
- If the least eigenvalue is -1, then the graph is the union of complete graphs
- If difference of the degree and the least eigenvalue is the number of vertices, then the graph is a **join**.
  Γ_G is formed by copies of smaller graphs that have all edges between them.

Cographs

Definition

A **cograph**, or complement-reducible graph, is a graph that can be generated from the single-vertex graph K_1 by complementation and disjoint union.

Example (Gap's TransitiveGroup (8,13))

Structure Description is $C_2 \times A_4$, its size is 24.

The spectrum of $G_{C_2 \times A_4}$ is: $\{15^{(1)}, \quad 3^{(4)}, \quad -1^{(18)}, \quad -9^{(1)}\}$

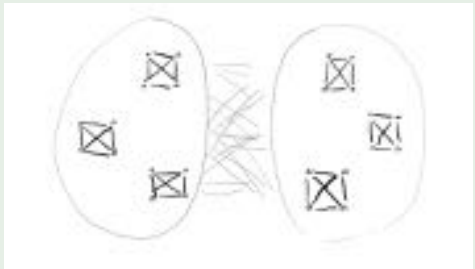
(Since $15 - (-9) = 24$, so the graph is a join.)

The complement has spectrum: $\{8^{(2)}, \quad 0^{(18)}, \quad -4^{(4)}\}$

(This is the disjoint union of two graphs with 12 vertices.)

The complement of one of these graphs has spectrum: $\{3^{(3)}, -1^{(9)}\}$

(This is the disjoint union of three copies of K_4.)



It is easy to spot a vertex-transitive cograph from its eigenvalues!

Delsarte-Hoffman Ratio Bound for Cocliques

A **weighted adjacency matrix** of a graph is a

- symmetric matrix with rows/columns indexed by the vertices of the graph;
- and the (v, w)-entry is 0 if v and w are not adjacent.

Theorem

Let A a weighted adjacency matrix for the graph X with constant row sum d. Then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where τ is the least eigenvalue of $A(X)$.

If equality holds then $\{S, V(X)\backslash S\}$ is an equitable partition: every vertex in $V(X)\backslash S$ is adjacent to exactly $-\tau$ vertices in S.

Willem H. Haemers wrote a nice paper on this bound [H].

Refinement the Derangement Graph.

For a conjugacy class C_i of G, define a graph X_{C_i} by

- the vertices elements of G,
- and σ, π are adjacent if $\pi^{-1}\sigma \in C_i$.

Set $A_{C_i} = A(X_{C_i})$, then

$$\Gamma_G = \bigcup_{C_i \text{ derangement}} X_{C_i}, \quad \text{and} \quad A(\Gamma_G) = \sum_{C_i \text{ derangement}} A_{C_i}$$

- $A_0 = I$ if $C_0 = \{id\}$ and $\sum_i A_{C_i}$ is the all ones matrix,
- For any i and j

$$A_i \; A_j = \sum_{\lambda} p_{i,j}^k A_k$$

- The algebra generated by all the A_i is commutative and spanned by the

$$\{A_i : C_i \text{ is a conjugacy class of } G\}$$

.

The adjacency matrices A_i form an **Association Scheme**.

Delsarte / Hoffman Ratio Bound

We know that the adjacency matrix for Γ_G is

$$A(\Gamma_G) = \sum_{C_i \text{ derangement}} A_{C_i}$$

We can make a weighted adjacency matrix for Γ_G by weighting the conjugacy class C_i by w_{C_i}:

$$M = \sum_i \mathbf{w_{C_i}} A_{C_i}.$$

- The eigenvalues of M are

$$\frac{1}{\chi(1)} \sum_i \mathbf{w_{C_i}}\ \chi(\sigma)|C_i|.$$

- This is an easy way to make weighted adjacency matrices for which we can still calculate the eigenvalues.
- Find values of $\mathbf{w_{c_i}}$ to optimize the ratio bound applied to M.
  set the least eigenvalue to be -1, and maximum the largest eigenvalue.

Eigenvalues

To get the best bound in the ratio bound we set this up as a linear programming problem:

$$\text{Maximize :} \quad d = \sum_i \mathbf{w_{C_i}} |C_i|,$$

$$\text{Subject to}$$

$$-1 \leq \frac{1}{\chi_1(1)} \sum_i \mathbf{w_{C_i}}\ \chi_1(\sigma)|C_i|$$

$$\cdots$$

$$-1 \leq \frac{1}{\chi_\ell(1)} \sum_i \mathbf{w_{C_i}}\ \chi_\ell(\sigma)|C_i|$$

This is very easy to solve for specific groups (I have been using Gurobi), and for families of groups for which the characters are well-understood.

Weightings for Derangement Graphs

This works for many objects:

1. t-intersecting sets (Wilson's proof from 1984)
2. t-intersecting vector spaces (and many other geometries)
3. Permutations
4. Perfect matchings
5. Designs

Theorem (M, Spiga, Tiep)

All two transitive groups have the EKR property.

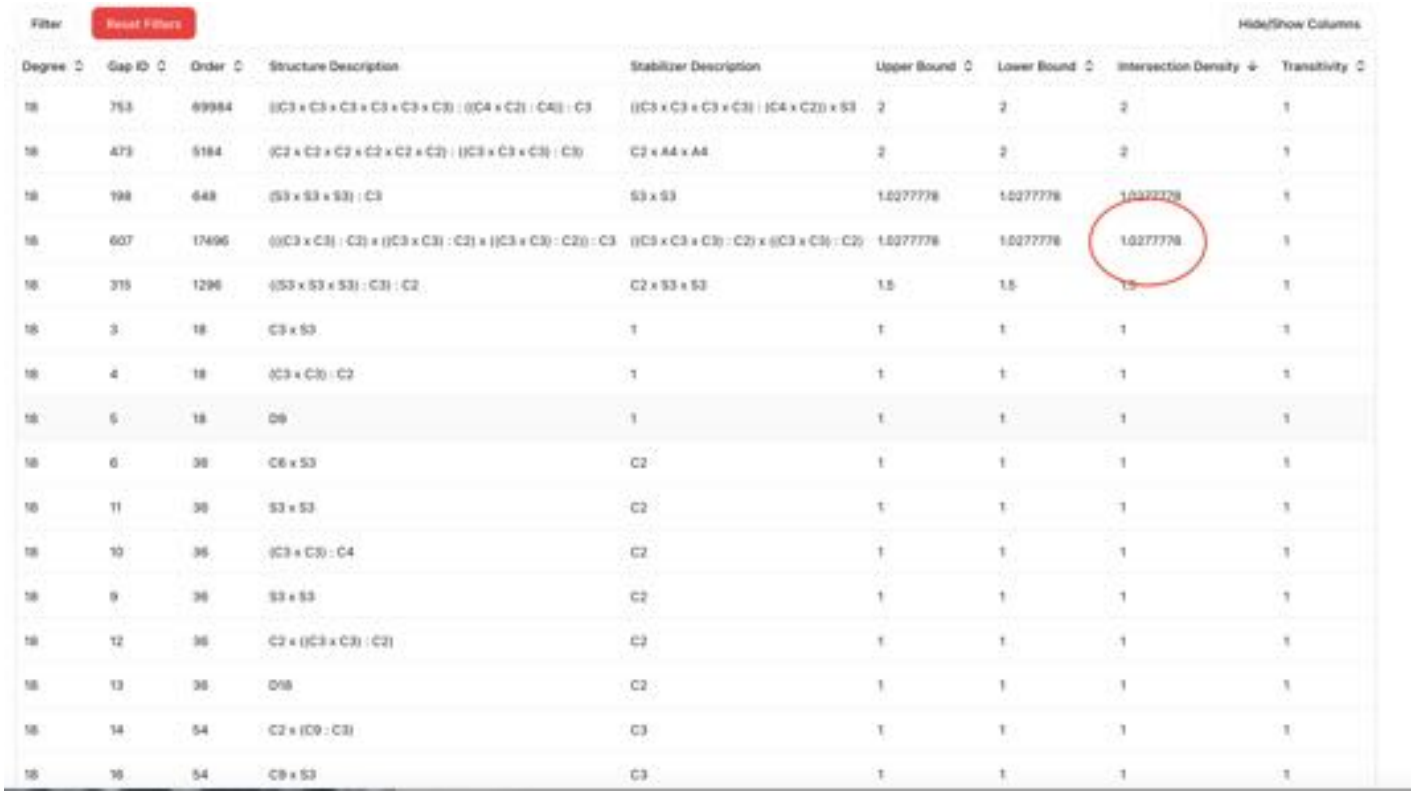
Use a graph homomorphism, classification of finite simple groups and weighting.

Regina can get pretty cold:



Data Base

Cody Antal, Cody Solie and I have a database of intersection density of small groups. The data base is at https://intersection-density.vercel.app/



Filter | Reset Filters | Hide/Show Columns

Degree	Gap ID	Order	Structure Description	Stabilizer Description	Upper Bound	Lower Bound	Intersection Density	Transitivity
18	753	69984	((C3 x C3 x C3 x C3 x C3 x C3) : ((C4 x C2) : C4)) : C3	(((C3 x C3 x C3 x C3) : (C4 x C2)) x S3	2	2	2	1
18	473	5184	(C2 x C2 x C2 x C2 x C2 x C2) : ((C3 x C3 x C3) : C3)	C2 x A4 x A4	2	2	2	1
18	198	648	(S3 x S3 x S3) : C3	S3 x S3	1.0277778	1.0277778	1.0277778	1
18	607	17496	(((C3 x C3) : C2) x ((C3 x C3) : C2) x ((C3 x C3) : C2)) : C3	(((C3 x C3 x C3) : C2) x ((C3 x C3) : C2)	1.0277778	1.0277778	1.0277778	1
18	315	1296	((S3 x S3 x S3) : C3) : C2	C2 x S3 x S3	1.5	1.5	1.5	1
18	3	18	C3 x S3	1	1	1	1	1
18	4	18	(C3 x C3) : C2	1	1	1	1	1
18	5	18	D9	1	1	1	1	1
18	6	36	C6 x S3	C2	1	1	1	1
18	11	36	S3 x S3	C2	1	1	1	1
18	10	36	(C3 x C3) : C4	C2	1	1	1	1
18	9	36	S3 x S3	C2	1	1	1	1
18	12	36	C2 x ((C3 x C3) : C2)	C2	1	1	1	1
18	13	36	D18	C2	1	1	1	1
18	14	54	C2 x (C9 : C3)	C3	1	1	1	1
18	16	54	C9 x S3	C3	1	1	1	1

The ratio bound applied to a weighted adjacency matrix is the most effective method.

Characterization

There is a second part to the Delsarte / Hoffman Ratio Bound:

Ratio Bound - second part

If equality holds in the ratio bound and S is a maximum coclique, then

$$\mathbf{1}_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is a τ-eigenvector (τ is the least eigenvalue).

The entries of $\mathbf{1}_S$ correspond to the objects and the entry is 1 is the object is in S and 0 otherwise.

1. $\mathbf{1}$ is the all ones vector, it is an eigenvector for d, the degree (largest eigenvalue)
2. Let U denote the span of the d-eigenspace and the τ-eigenspace.
3. If equality holds in ratio bound, then for S any maximum coclique $\mathbf{1}_S \in U$.
4. In many cases U is spanned by the characteristic vectors of the canonical cocliques.

Characterization

In many cases where equality holds in the ratio bound, the characteristic vector of any maximum coclique is a linear combination of characteristic vectors of characteristic vectors of canonical cocliques.

Filmus and Lindzey [FL] give a nice method to use this fact to get a characterization. Let S be a maximum coclique, where the ratio bound holds with equality.

1. Let E_λ be the projection to the λ eigenspace.
2. If P is the projection to U, then $P = E_\tau + E_d$,
3. If S is a maximum coclique, then $\mathbf{1}_S = P\mathbf{1}_S$.
4. $S_{\downarrow i}$ is the set of objects in S that contain i.
   part of S that is contained in a canonical coclique.
5. It is not difficult to calculate the projection of $\mathbf{1}_S$ to the τ-eigenspace is terms of $|S_{\downarrow i}|$.

Characterization

1. Assume there is an $x \in S$ and $y \notin S$ where x and y differ in exactly one position
2. $P(\mathbf{1}_S)[x] = 1$ and $P(\mathbf{1}_S)[y] = 0$.
3. We picked x and y to differ in only one atom, $j \in x \backslash y$ and $k \in y \backslash x$ so there is a constant
$$a = |S_{\downarrow j}| - |S_{\downarrow k}|$$
4. $S \backslash S_{\downarrow i}$ is bounded above, since it is intersecting on a smaller base sets; so $S_{\downarrow i}$ is large.
5. $S_{\downarrow i}$ is so large that $S_{\downarrow i} = S$.

This can be used in many cases to characterize all maximum intersecting sets.

Robustness of EKR results

"Stability versions of such theorems assert that if the size of a family is close to the maximum possible size, then the family itself must be close (in some appropriate sense) to a maximum-sized family." [EKL]



- For objects that have the EKR property, what is the largest intersecting family that is **not contained in a canonical** intersecting family?
  These are called Hilton-Milner Theorems or stability theorems [HM]
- We can consider **all** cocliques in a derangement graph, what proportion are subsets of a canonical set?
- Consider **almost intersecting**—for each element in the family there is at most one other element that it doesn't intersect.

Robustness of the EKR theorem

If Γ is the derangement graph for some object that has the EKR property, can we randomly remove edges, without creating a larger coclique with high probability?

- Keep the edges from the Kneser graph $K(n,k)$ probability is more than $\frac{\ln(n\binom{n-1}{k})}{\binom{n-k-1}{k-1}}$ then the cocliques don't get any bigger with high probability. [BBN, BKL, TD]
  this is the probability of forming a coclique by adding a single vertex to a maximum coclique.
- For perfect matchings and permutations an analogous result holds [GMMPS].

Random Subgraphs of the Derangement Graph

Let $\Gamma_{n,p}$ be the random spanning subgraph of the derangement graph Γ_n where each edge is included independently with probability p.

- Probability of S being a coclique in $\Gamma_{n,p}$ is $(1-p)^{\mathcal{E}(S)}$, where $\mathcal{E}(S)$ is number of edges in S.
- Bounds on the size of $\mathcal{E}(S)$ are known as *isoperimetry* results,

Easiest way to make a larger coclique

1. The largest coclique $S_{i,j}$ is a “star"; all permutations that map $i \to j$.
2. This coclique meets the ratio bound with equality.
3. By ratio bound, a vertex outside the star is adjacent to exactly $-\tau$ vertices in the star.
4. $\mathcal{E}(\{S_{i,j} \cup v\}) = -\tau$;
5. so the probability that $\{S_{i,j} \cup v\}$ is a coclique in $\Gamma_{n,p}$ is $(1-p)^{-\tau}$

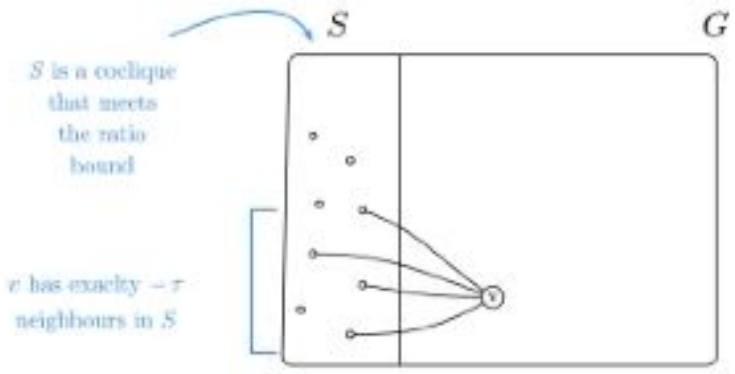
If all these τ edges are removed in $\Gamma_{n,p}$, then $\{S_{i,j} \cup v\}$ is a coclique called **a super star**, and denoted $\mathcal{S}_{i,j}[v]$.

Regina has superstars with high probability:





Easiest Way to make a Larger Coclique



Lemma

The expectation that there will be a superstar in $\Gamma_{n,p}$*:*

$$n^2 \quad \left(n! - (n-1)!\right) \quad (1-p)^{-\tau} \quad < \quad n^2 \quad \left(n! - (n-1)!\right) \quad (e^{-p})^{-\tau}$$

Set the probability so the expectation is 1, so $p_c = \dfrac{\ln\left(n!\, n(n-1)\right)}{\tau}$.

The least eigenvalue equals $\tau = d_n/(n-1)$, for the upper bound, use $1 + x \leq e^x$ if $|x| \leq 1/2$.

Superstars

Theorem

Let d_n be the number of derangements in $\mathrm{Sym}(n)$. Let

$$p_c(n) := \frac{(n-1)\ln(n!\, n(n-1))}{d_n}.$$

Then, for any $\varepsilon > 0$, as $n \to \infty$,

$$\mathbb{P}\left[\alpha(\Gamma_{n,p}) = (n-1)!\right] \to \begin{cases} 1 & \text{if } p \geq (1+\varepsilon)p_c(n), \\ 0 & \text{if } p \leq (1-\varepsilon)p_c(n). \end{cases}$$

Furthermore, when $p \geq (1+\varepsilon)p_c(n)$, with high probability, every maximum independent set in $\Gamma_{n,p}$ is a star.

1. If $p \geq (1+\varepsilon)p_c(n)$, there will not be a superstar—or any other kind of large coclique—with high probability.
2. If $p \leq (1+\varepsilon)p_c(n)$, then independent superstars appear in $\Gamma_{n,p}$, with high probability.

Proof Outline

1. For a probability p, set Y to be the probability that there is a superstar in $\Gamma_{n,p}$.
2. From Chebyshev's inequality, $\mathbb{P}[Y > 0] \leq \mathbb{E}[Y]$.
3. We picked $p \geq (1+\varepsilon)p_c(n)$, so that $\mathbb{E}[Y]$ goes to 0 as n goes to infinity.

Lemma

If $p \geq (1+\varepsilon)p_c$, then

$$\lim_{n\to\infty} \mathbb{P}[Y > 0] = 0.$$

If p larger than p_c, don't have superstars with high probability.

Proof Outline

1. The events $\{\mathcal{S}_{i,j}[v] \text{ is independent}\}$ are mutually independent across all choices of $v \notin \mathcal{S}$.

$$\begin{aligned}\mathbb{P}\left[\alpha(\Gamma_{n,p}) = \alpha(\Gamma_n)\right] &\leq \mathbb{P}\left[\mathcal{S}_{i,j}[v] \text{ is not independent for all } v \notin \mathcal{S}\right] \\ &\leq \prod_{v \notin \mathcal{S}} \left(1 - \mathbb{P}\left[\mathcal{S}_{i,j}[v] \text{ is independent}\right]\right) \\ &\leq \left(1 - (n^2(n! - (n-1)!))^{-(1-\varepsilon/2)}\right)^{n!-(n-1)!} \\ o(1).\end{aligned}$$

2. This goes to 0 as n is large.

Lemma

If $p < p_c$, independent superstars appear in $\Gamma_{n,p}$ with high probability.

Faux Stars

1. A coclique $\mathcal{A}$ is called a *faux star* if it is larger than a star, but doesn't contain a star.
2. Pick a star $\mathcal{S}_x$ with maximum intersection with $\mathcal{A}$, the **type** of $\mathcal{A}$ is that size of star without $\mathcal{A}$.
3. Set X_i variable that there is an independent faux star of type i.
4. Consider the stars with i large and small separately.

Estimate the number of faux stars of type i, count the number of edges in the faux star in Γ_n.

For this we need some isoperimetry results.

Isoperimetry Results

From Das and Tran [DT], there is a lower bound on edge density.

Lemma

Let $G = (V, E)$ be a

1. *connected k-regular graph,*
2. *with smallest eigenvalue τ, and*
3. *second smallest eigenvalue μ.*

For eigenvalue ξ, let V_ξ be the corresponding eigenspace and $U = V_k \oplus V_\tau$.

For any $S \subset V$

$$\frac{2\mathcal{E}(S)}{|V|} \geq (k-\tau)\frac{|S|^2}{|V|^2} + \tau\frac{|S|}{|V|} - (\tau-\mu)\left\|\mathbf{1}_S - \mathrm{Proj}_U(\mathbf{1}_S)\right\|^2.$$

Where $|\cdot|$ is usual ℓ_2-norm.

Robustness of the EKR theorem in a Cayley Graph

For a permutation group G the derangement graph is

$$\Gamma_G = Cay(G, \mathrm{Der}(G)).$$

Removing random edges from Γ_G produces a graph that isn't regular, and not vertex transitive.

What happens if we make a new Cayley graph from a derangement graph by randomly remove vertices from the connection set $\mathrm{Der}(G)$? [GMMPS2]

Question

How many elements can be removed from $\mathrm{Der}(G)$ so that $\Gamma_G = Cay(G, \mathrm{Der}(G))$ without making a larger coclique?

In many examples, the size of the cocliques stay the same or doubles (so two star become a coclique).



Thanks!

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A Few References

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