Covering Arrays and Related Problems in Extremal Combinatorics

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University of Ottawa

What is a Covering Array?

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Testing Applications

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To do all combinations we need to have
$3 \times 3 \times 3 \times 3 \times 3 = 243$ parties

Testing the Perfect Party

Drinks	Snacks	Attire	Guests	Time
Beer	Veggies	Casual	Everybody	Whenever
Beer	Olives	Costume	Math Kids	7:30
Beer	Tofu	Formal	Girls Only	10:00
Wine	Veggies	Costume	Girls Only	7:30
Wine	Olives	Casual	Math Kids	10:00
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Covering Arrays on Graphs

A *covering array on a graph* G denoted $CA(n, G, g)$, has:

- $k = |V(G)|$ rows of length n (n is the size)
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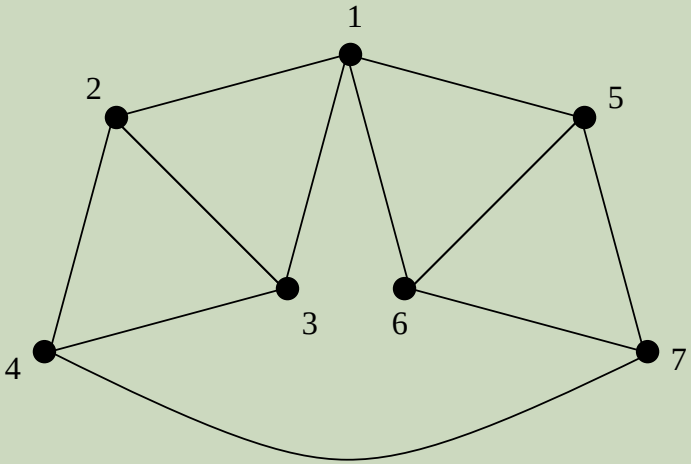

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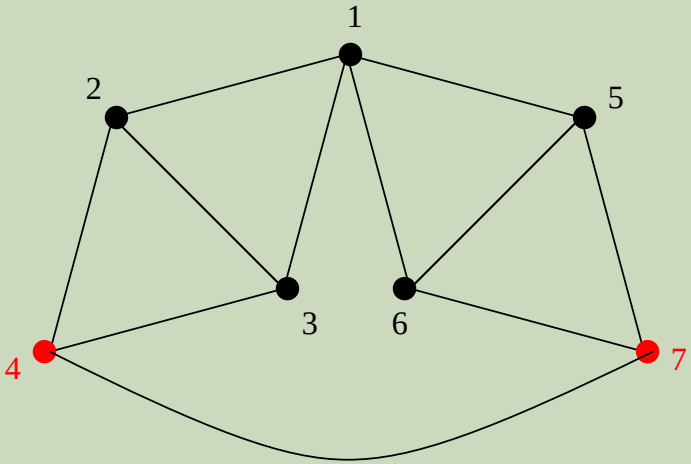

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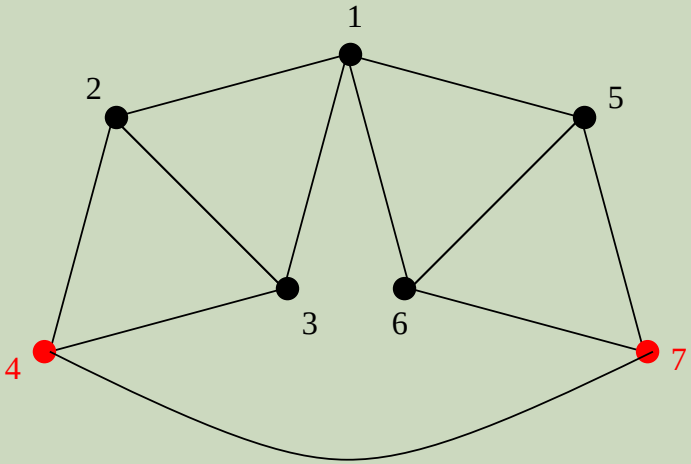

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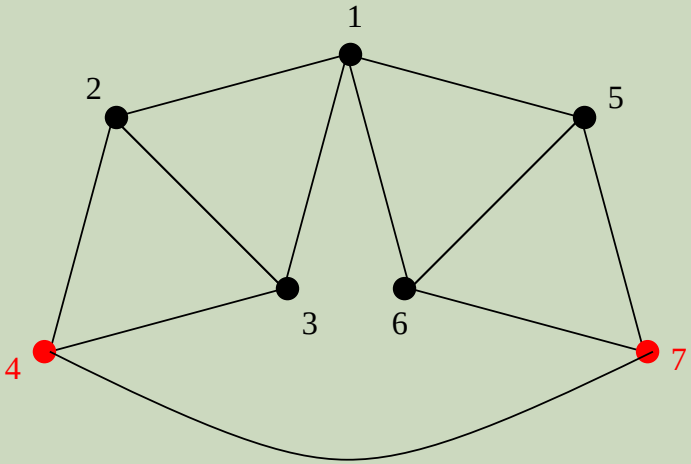

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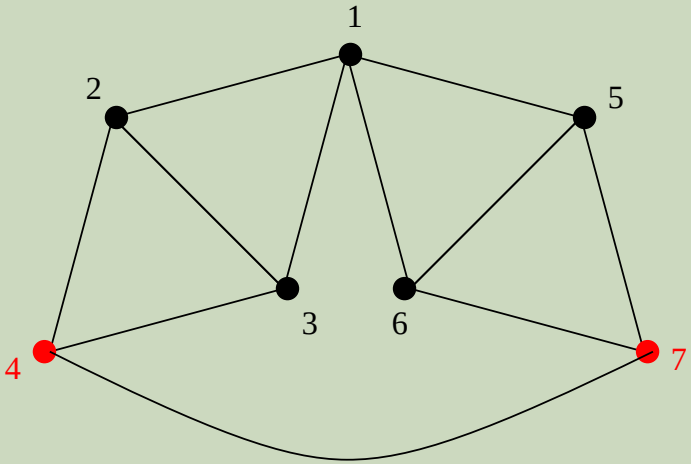

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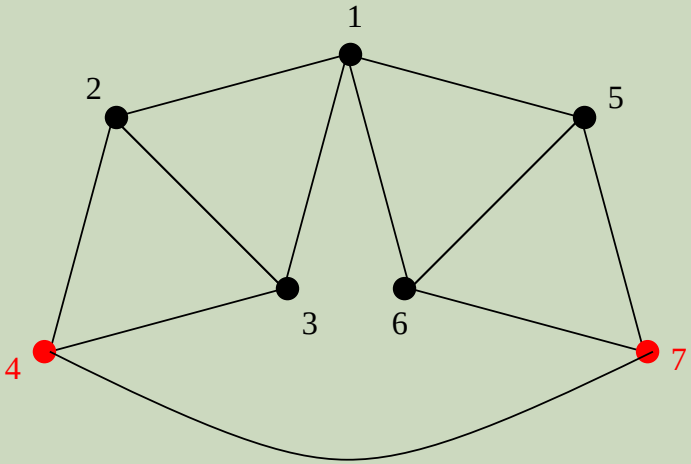

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A homomorphism, $f : G \to H$ is

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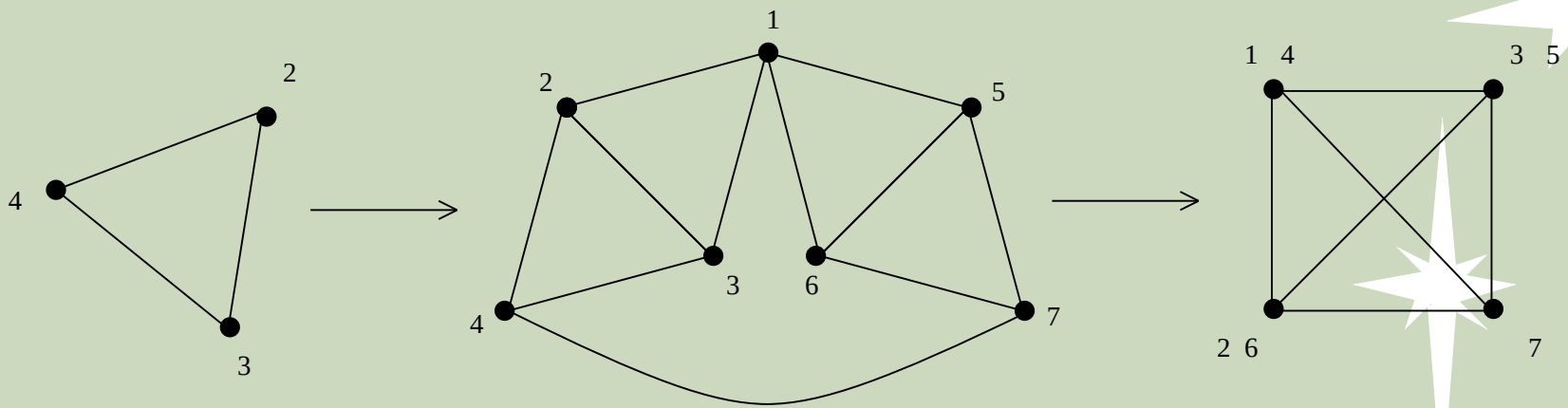

Motivation and Direction

- Normal covering arrays correspond to covering arrays on complete graphs

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$$CAN(\omega(G), g) \leq CAN(G, g) \leq CAN(\chi(G), g).$$

- and a bound from the chromatic number.

Qualitative Independence Graph

Define graph $QI(n, g)$ by

- Vertices are all vectors which could go into a row of a covering array of size n on $\mathbb{Z}_g$.
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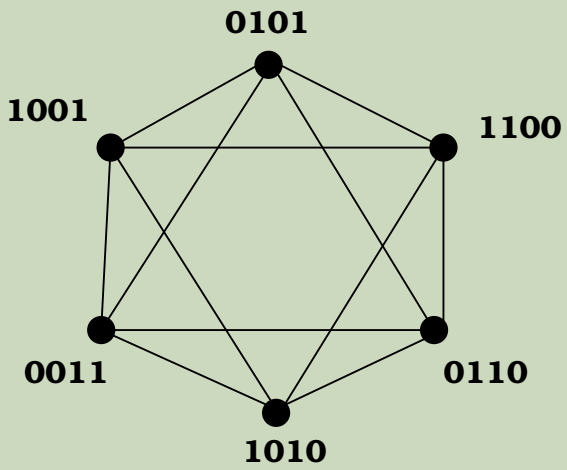

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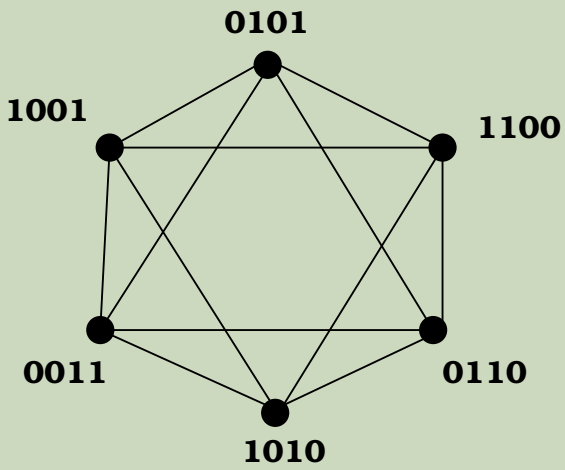

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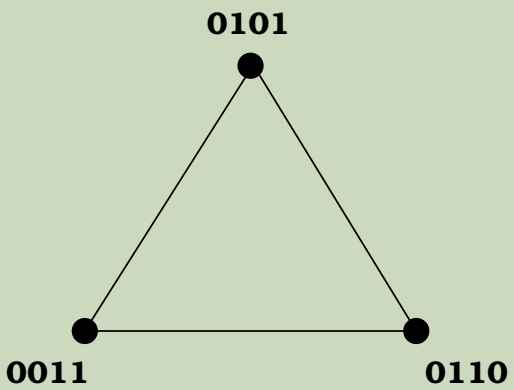

The graph $QI(4, 2)$.

Why is $QI(n, g)$ so Good?

This family determines up to homomorphism the covering number for a graph.

Theorem (M. and Stevens, 2002)
A $CA(n, G, g)$ exists if and only if there is a homomorphism

$$G \to QI(n, g).$$

Parallel theorems!

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About $QI(4,2)$

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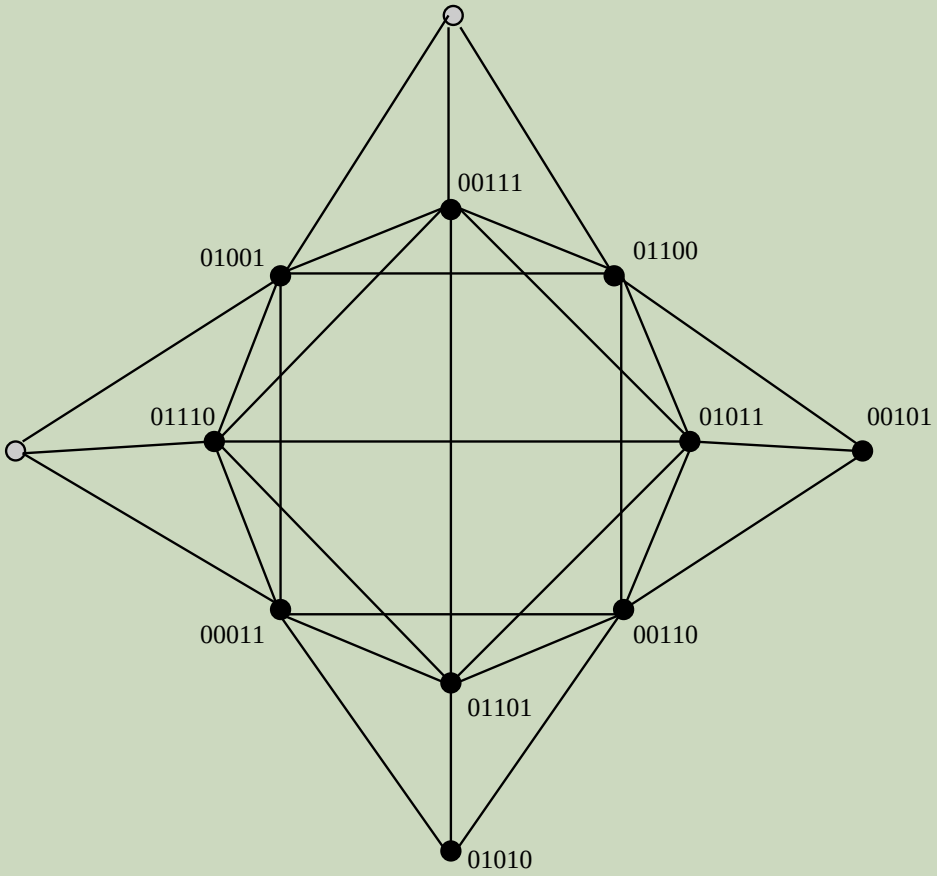
Determining if a graph has a binary covering array of size 4 is the same as determining if a graph is 3-colourable (NP-complete, Seroussi and Bshouty, 1988).

The Graph $QI(5,2)$



Interesting facts about this graph

- $V(QI(5,2)) = 10$
- $\omega(QI(5,2)) = 4$
- $\chi(QI(5,2)) = 5$

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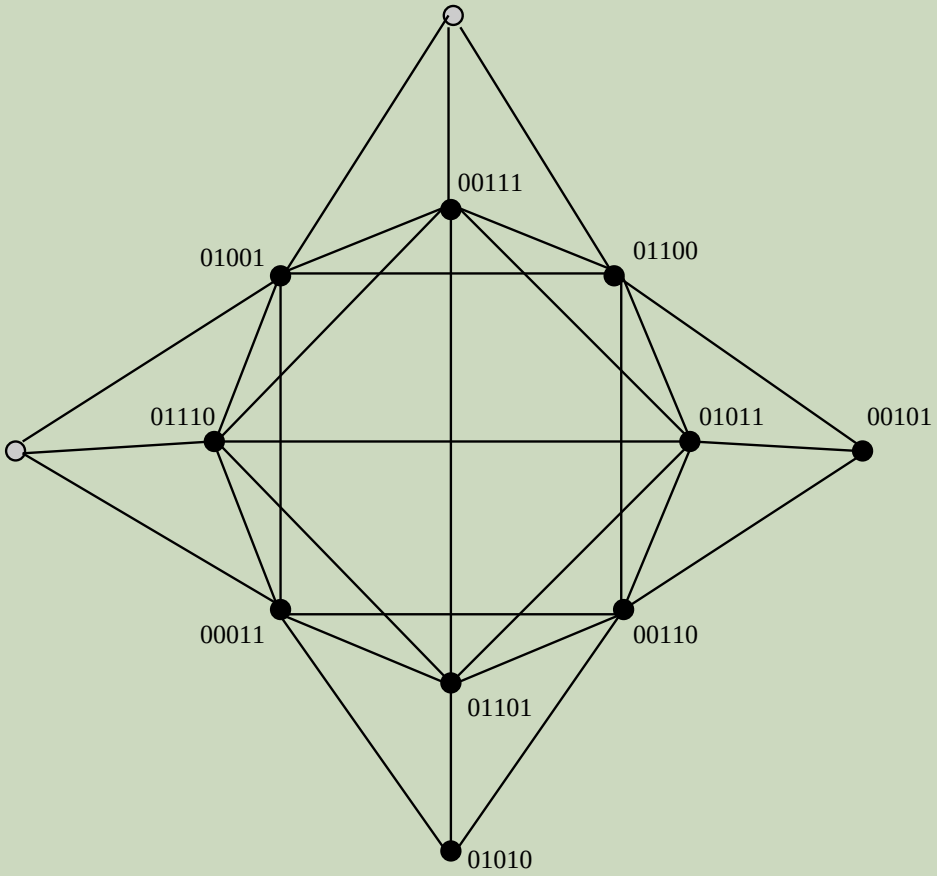

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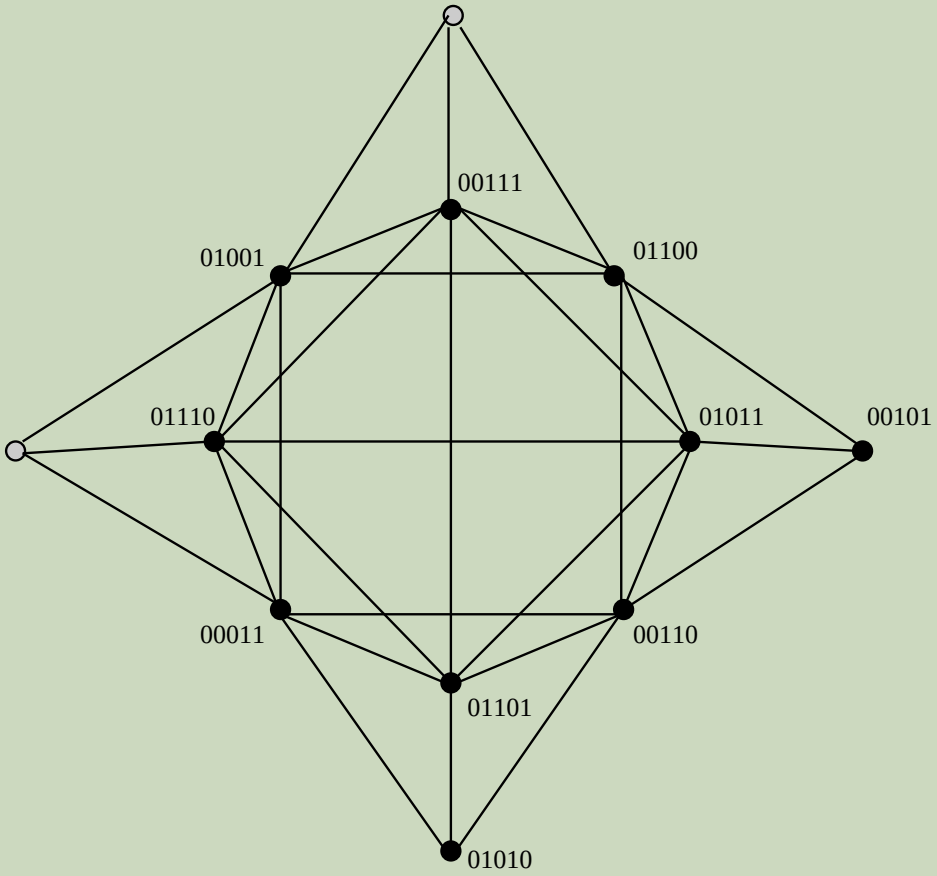

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$$CAN(QI(5,2),2) = 5 < 6 = CAN(5,2).$$

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Theorem (Kleitman and Spencer 1973, Katona 1973)

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Theorem (M. and Stevens 2002)

For n even $\quad CAN(QI(n,2),2) = CAN(\chi(QI(n,2)),2)$,

and n odd $\quad CAN(QI(n,2),2) = CAN(\chi(QI(n,2)),2) - 1$.

Connection to Set Systems

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		{3,4,5}			and		{2,		4,5}		and			{3,4}		

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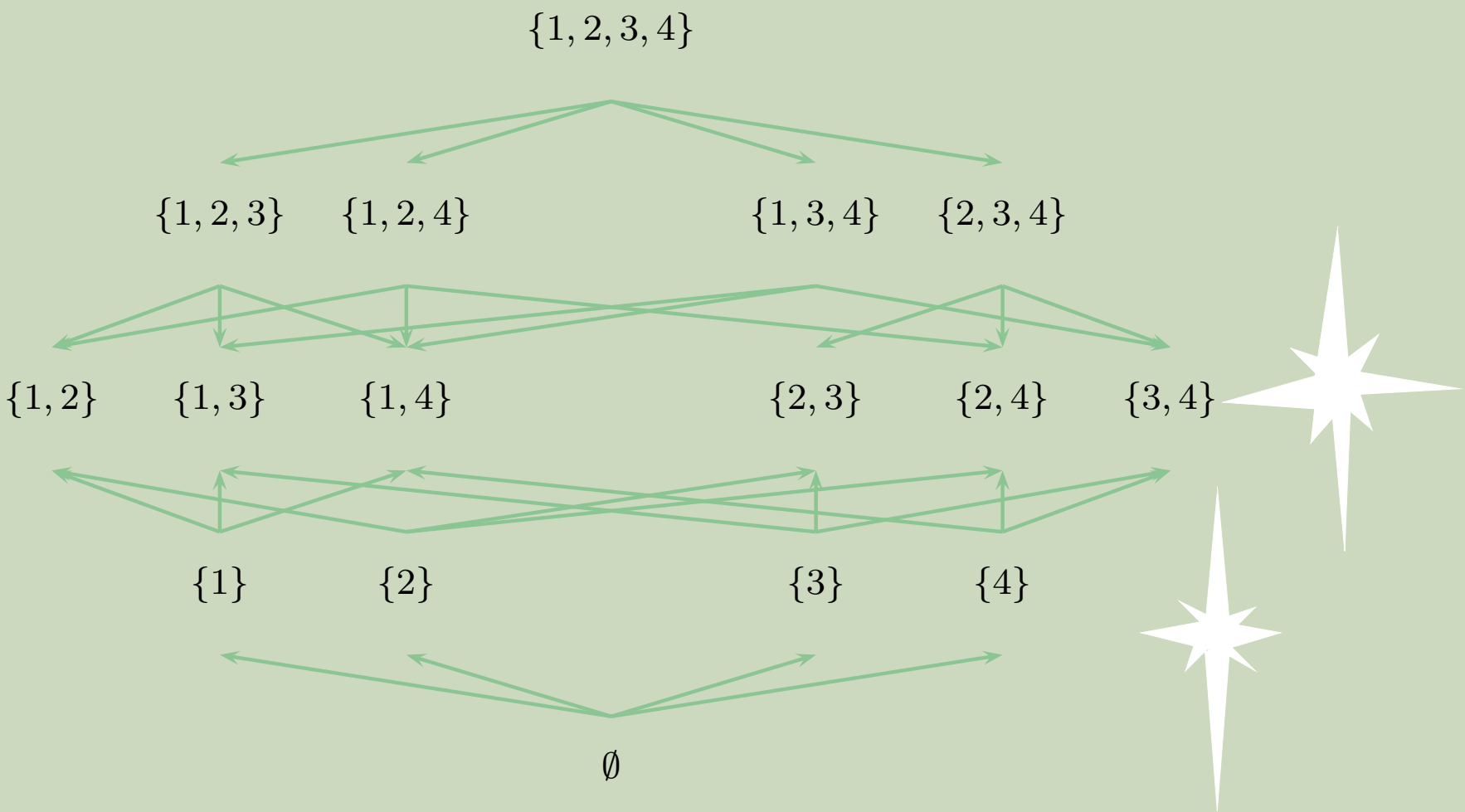
To be qualitatively independent sets are

- intersecting
- not contain each other

Mapping to the Core



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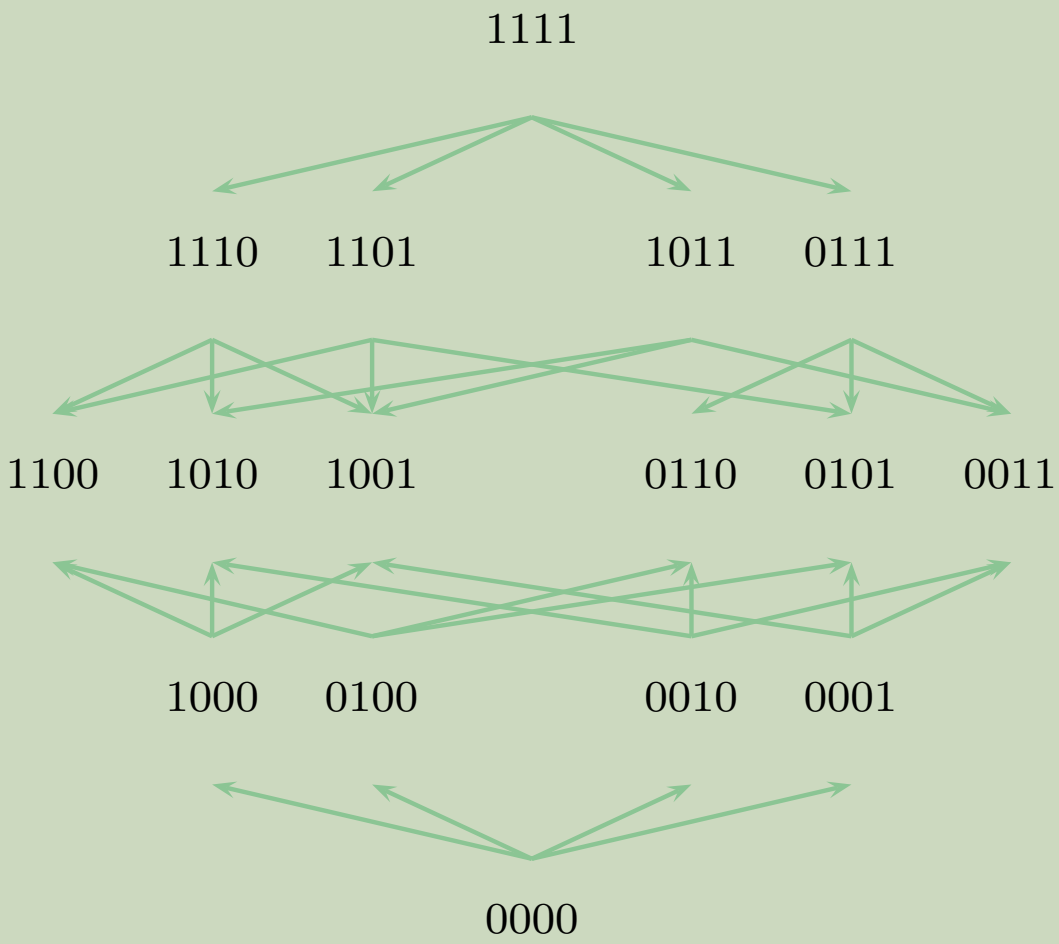

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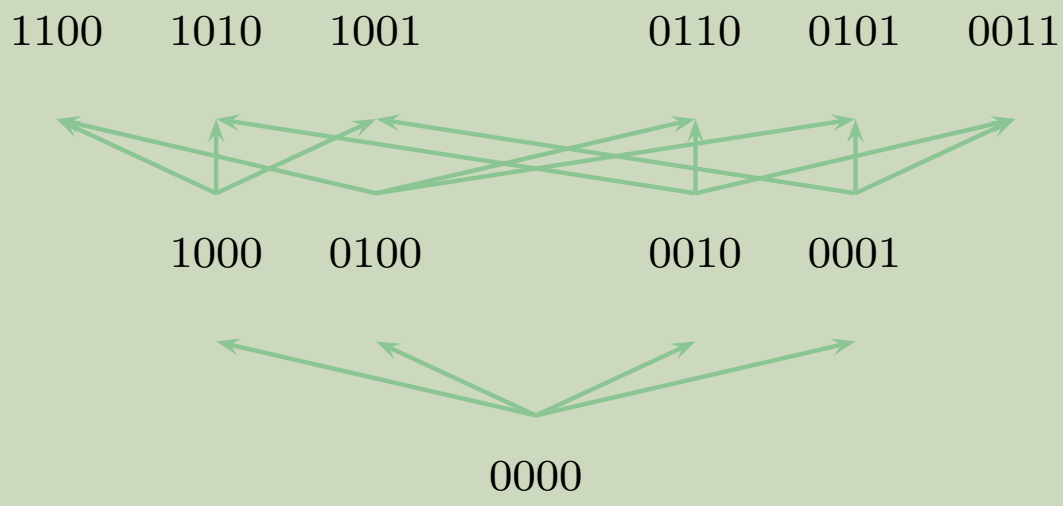
- consider vectors with less than half 1's



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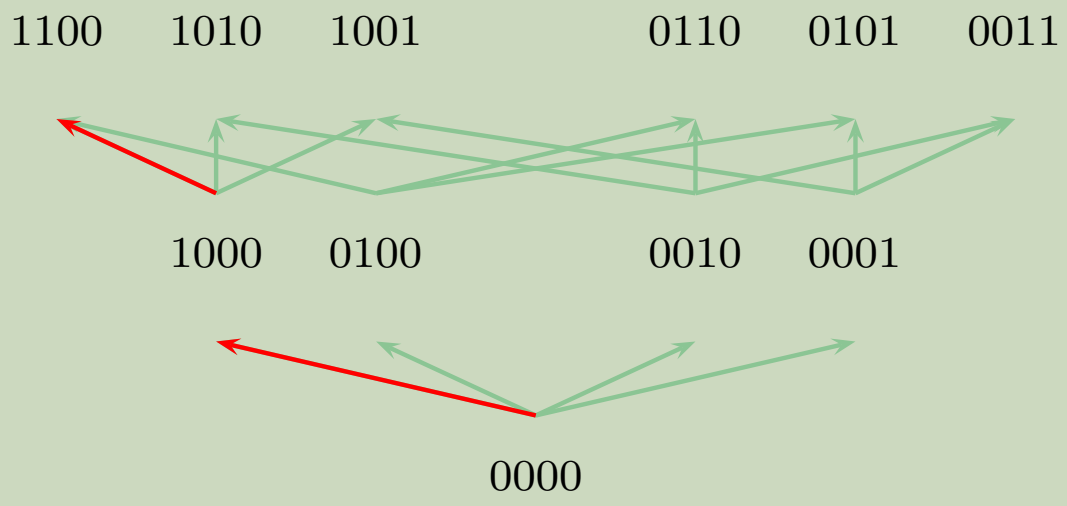

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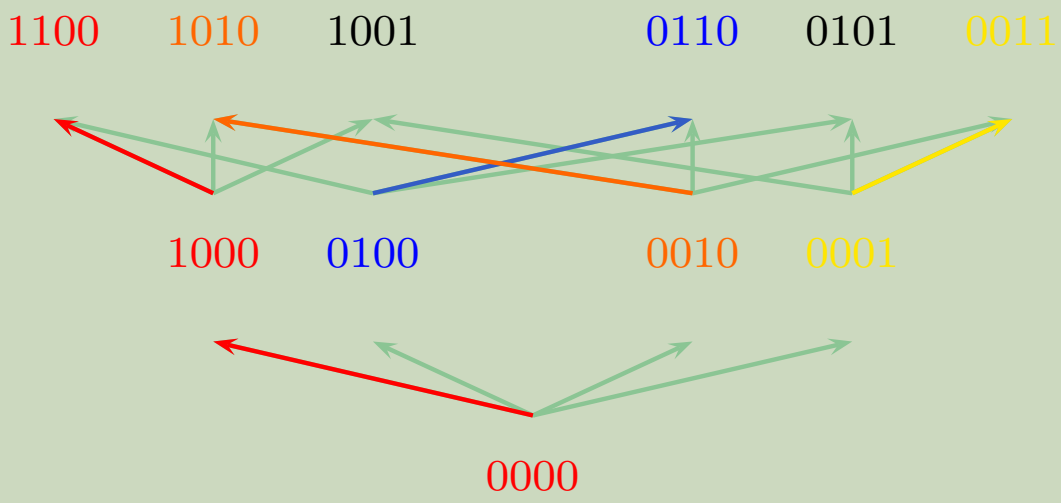

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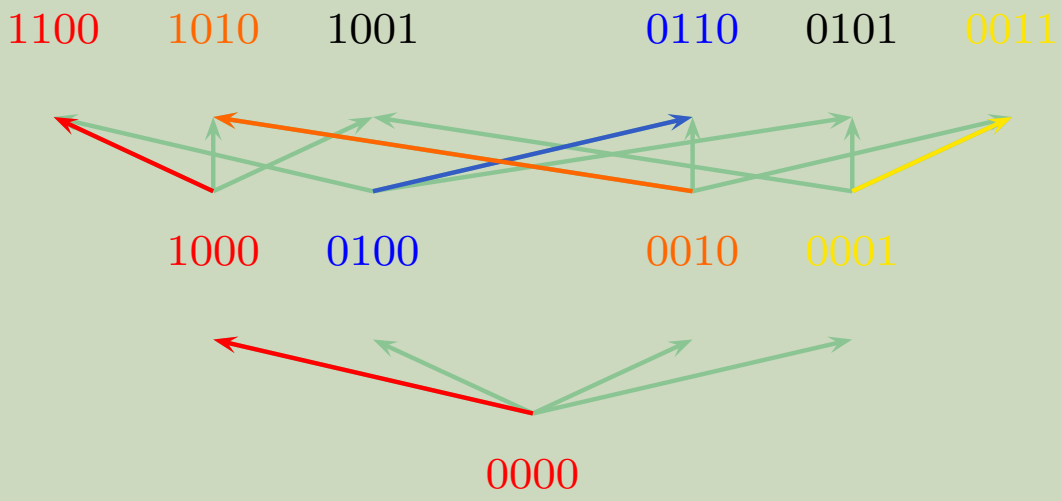

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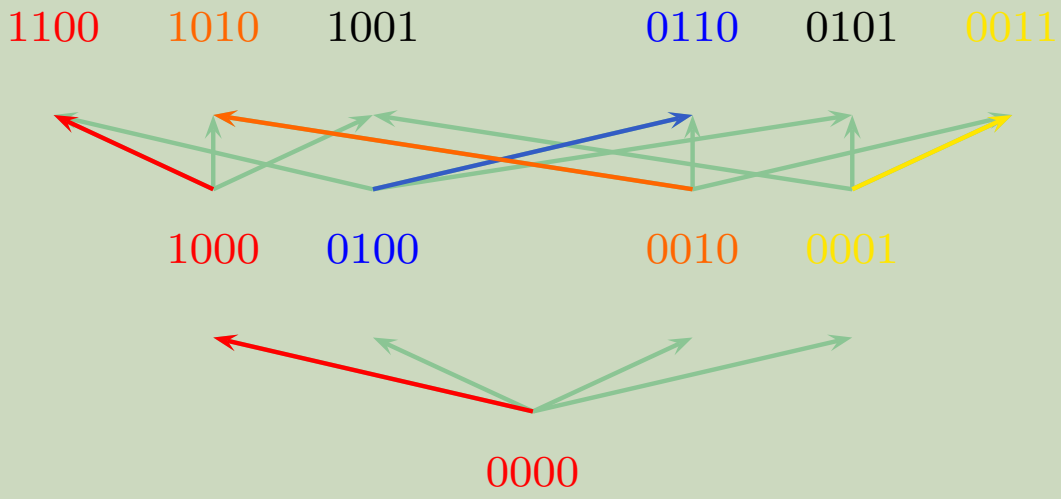

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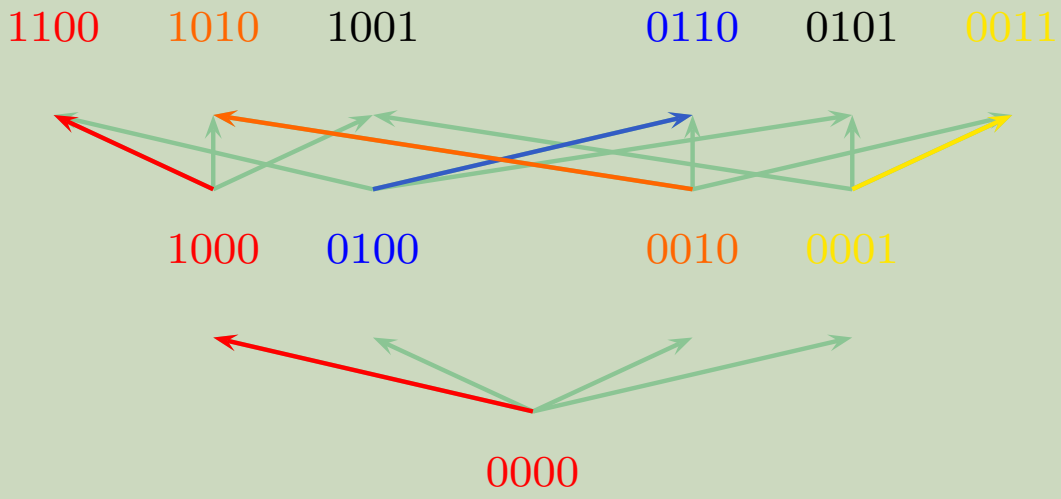

- use chains to map to $\lfloor n/2 \rfloor$ level
- map preserves qualitative independence
- graph with vertices from $\lfloor n/2 \rfloor$ level is core

The Core of $QI(n,2)$

Theorem (M. and Stevens 2002)
For n even the core of $QI(n,2)$ is $K_{\frac{1}{2}\binom{n}{\frac{n}{2}}}$.

For n odd the core is the subgraph induced by vertices with $\frac{n-1}{2}$ 1's.

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We can just use the vertices of $QI(n,2)$ which correspond to $\lfloor n/2 \rfloor$ sets of n

Cliques

A maximum system of intersecting $\lfloor n/2 \rfloor$-sets of n is a maximum clique in $QI(n, 2)$.

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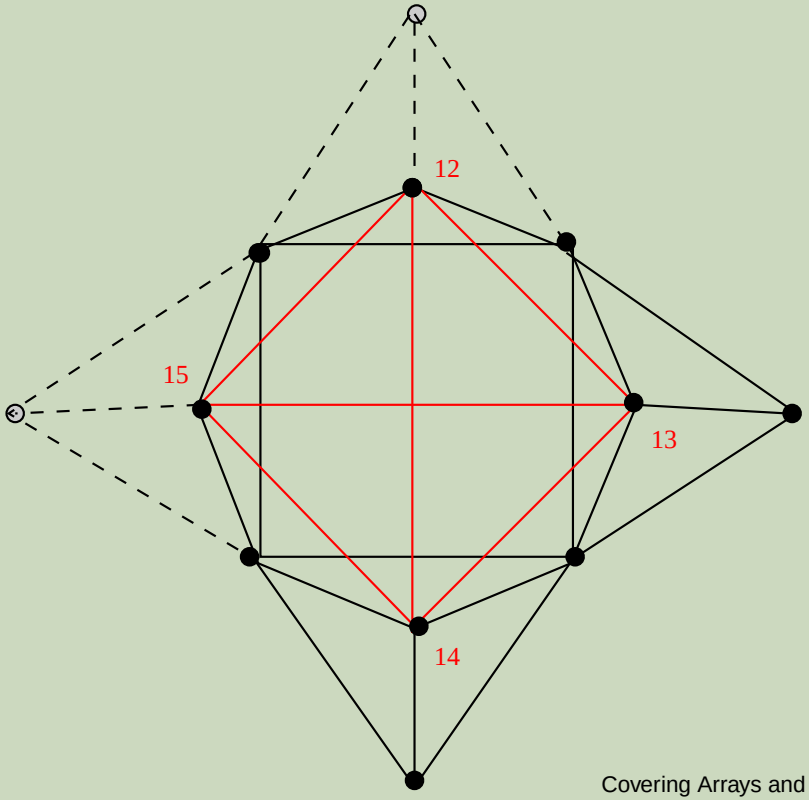

The Erdős-Ko-Rado theorem

Intersecting Set System A k-set system with all sets pairwise intersecting.

123	124	125	126	134
135	136	145	146	234

The Erdős-Ko-Rado theorem

Trivial Intersecting Set System All sets have a common element

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For $n = 6$ and $k = 3$ size is $\binom{5}{2} = 10$

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For $n = 6$ and $k = 3$ size is $\binom{5}{2} = 10$

Theorem (Erdős-Ko-Rado 1961) For $n \geq 2k$, the maximal intersecting k-set system is a trivial system, if $n > 2k$, this maximum is unique.

This solves $CAN(k,2)$

Corollary $\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}$.

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A clique in $QI(n,g)$ is a set of qualitatively independent vertices.

A $CA(n,k,g)$ exists iff $\omega(QI(n,g)) \geq k$

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Corollary $CAN(k,2) = \min\{n \; ; \; k \leq \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}\}$.

This almost solves $CAN(G, g)$

For any graph G

$$CAN(\chi(G), 2) - 1 \leq CAN(G, 2) \leq CAN(\chi(G), 2).$$

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For any graph G

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We have necessary conditions on G for

$$CAN(G, 2) < CAN(\chi(G), 2).$$

These are:

- $CAN(\chi(G), 2) = c$ must be even, and
- $\binom{c-2}{\frac{c}{2}} < \chi(G) \leq \left\lceil \frac{1}{2}\binom{c-1}{\frac{c}{2}} \right\rceil.$

Beyond the Binary

In the general case, the vertices of $QI(n, g)$ are like g-ary partitions.

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1 2 3 | 4 5 6 | 7 8 9	1 4 7 | 2 5 8 | 3 6 9

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- What would intersecting partitions correspond to in $QI(n, g)$?

Uniform Partitions

In $QI(g^2, g)$ partitions correspond to adjacent vertices if no pairs in any class are repeated:

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Uniform Partition A partition which has all classes the same size.

t-Partially Intersecting

Partitions

$$A = \{A_1, A_2, \ldots, A_g\} \quad B = \{B_1, B_2, \ldots, B_g\}$$

are t-*partially intersecting* if there exist an i and a j so that

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A *trivial* 2-partially intersecting system

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⋮		
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Conjecture $\alpha(QI(g^2,g)) = \binom{g^2-2}{g-2}\frac{((g-1)g)!}{(g-1)!(g!)^{(g-1)}}$.

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Conjecture Any maximum intersecting uniform partition system is trivial.

Theorem (M. and Moura 2004) Let $g \geq 1, c \neq 2$ and $n = gc$. A largest intersecting uniform g-partition system must be trivially intersecting.

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