Derangement Graphs of Groups

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Permutations

Definition

Two permutations $\sigma, \pi \in \mathrm{Sym}(n)$ **agree or intersect** if for some $i \in \{1, 2, \ldots, n\}$

$$i^\sigma = i^\pi.$$

1. Two permutations σ and π intersect if and only if $\pi^{-1}\sigma$ has a fixed point.
2. A permutation is a **derangement** if it fixes no points.
3. Permutations σ and π are intersecting if and only if $\pi^{-1}\sigma$ is **not** a derangement.

Definition

A set of permutations is **intersecting** if any two elements from the set are intersecting.

What is the largest set of intersecting permutations in a permutation group?

(This depends on group action!)

Canonical Intersecting Set

- The stabilizer of a point in a group G is an intersecting set

$$G_i = \{\sigma \in G \,|\, i^\sigma = i\}.$$

- Any coset of a stabilizer of a point is an intersecting set

$$S_{i,j} = \{\sigma \in G \,|\, i^\sigma = j\}.$$

  These are called the **canonical intersecting sets**.

- If G is transitive of degree n, then $|S_{i,j}| = \frac{|G|}{n}$.

Lemma

For any transitive group G with degree n, there is an intersecting set of size $\frac{|G|}{n}$.

Erdős-Ko-Rado Theorem

Theorem (Erdős-Ko-Rado Theorem - 1961)

Let $\mathcal{A}$ be an intersecting k-set system on an n-set. If $n > 2k$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

1. The largest intersecting collection of intersecting k-sets is the collection of all sets that contain a common point.
2. Sometimes such a set is called **trivially intersecting**,
3. or **canonically** intersecting.

Derangement Graph

Definition

For any permutation group G we can define a **derangement graph**, Γ_G.

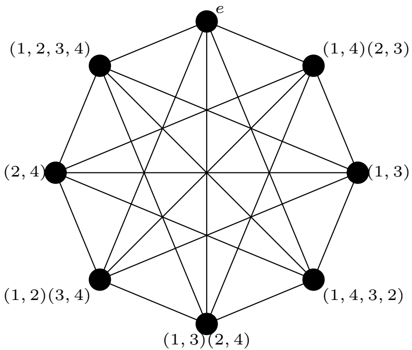
- The vertices are the elements of G.
- Vertices $\sigma, \pi \in G$ are adjacent if and only if $\pi^{-1}\sigma$ is a derangement.

(So permutations are adjacent if they are **not** intersecting.)

An intersecting set in G is a coclique (independent set) in Γ_G, if G be a transitive group with degree n then $\alpha(\Gamma_G) \geq \frac{|G|}{n}$.

- The derangement graph is **regular**, the degree is the number of derangements in the group.
- G is a subgroup of the automorphism group of Γ_G.
- This graph is **vertex transitive** the automorphism group acts transitively on the vertices.

Derangement Graph



The graph $\Gamma_{D(4)}$.

Intersection Density

Let $G \leq \mathrm{Sym}(n)$ be any finite transitive group.

1. For $\mathcal{F} \subseteq G$ intersecting, define the **intersection density of** $\mathcal{F}$ to be

$$\rho(\mathcal{F}) = |\mathcal{F}| \left(\frac{|G|}{n} \right)^{-1} = \frac{|\mathcal{F}|}{|G_x|}.$$

2. The intersection density of the stabilizer of a point is 1.
3. The **intersection density of the group** G is

$$\rho(G) := \max\{\rho(\mathcal{F}) \,|\, \mathcal{F} \subseteq G \text{ is intersecting}\}.$$

(This was defined by Li, Song and Pantangi in 2020.)

Observation

The intersection density of any transitive permutation group is at least 1.

Groups with intersection density 1 are exactly the groups that have the Erdős-Ko-Rado Property.

Questions about Intersection Density

Which transitive groups have intersection density 1?

These groups are said to the have the EKR property

How large can the intersection density be?

Example

There is a group of order 12, acting on a set of size 6.

$$\{(), (1,4)(2,5), (2,5)(3,6), (1,4)(3,6),$$
$$(1,2,3)(4,5,6),\ (1,3,2)(4,6,5),\ (1,2,6)(3,4,5),\ (1,6,2)(3,5,4),$$
$$(1,3,5)(2,4,6),\ (1,5,3)(2,6,4),\ (1,5,6)(2,3,4),\ (1,6,5)(2,4,3)\}$$

The stabilizer of point has size 2, and there is an intersecting set of size 4. This group has intersection density $4/2 = 2$.

Previous Results

The following groups have intersection density 1

1. $\mathrm{Sym}(n)$ with its natural action on $[1, \dots, n]$.
   (Frankl and Deza, 1977)
2. $\mathrm{Sym}(n)$ with its action on t-tuples, if n is sufficiently large.
   (Ellis, Friedgut, Pilpel, 2011)
3. $\mathrm{Sym}(n)$ with its action on t-sets, if n is sufficiently large.
   (Ellis, 2012). Conjectured to hold for all n.
4. $\mathrm{PGL}(n, k)$ on projective space.
   (Spiga and Meagher 2011, 2014; and Spiga 2016).
5. Frobenius groups.
   (Ahmadi and Meagher 2015)

Theorem (Meagher, Spiga, Tiep, 2016)

All 2-transitive groups have intersection density 1.

Intersecting Subgroups

Lemma

Let H be a permutation group. Then H is intersecting if and only if it is derangement free.

Proof. 1

If H is intersecting, then each $h \in H$ intersects the identity element, and hence has a fixed point.

Conversely, if H is derangement free, then for any $g, h \in H$ the element gh^{-1} is in H, so is not a derangement. Thus g and h are intersecting.

Corollary

If G is transitive of degree n with a derangement-free subgroup H, then

$$\rho(G) \geq \frac{|H|}{\frac{|G|}{n}} = \frac{n}{[G:H]}.$$

Group Actions

1. For any group G acting on a set Ω take H to be the stabilizer of a point. Then the action is equivalent to G acting on G/H.
2. If $g, h \in G$ are intersecting under this action, then for some $x \in G$

$$g(xH) = h(xH) \quad \Leftrightarrow \quad h^{-1}gxH = xH \quad \Leftrightarrow \quad x^{-1}h^{-1}gxH = H$$

3. So gh^{-1} is conjugate to an element of H.

We are looking for a set S of elements from G such that for any two $g, h \in S$ $h^{-1}g$ is conjugate to an element in H.

Example (Hujdurović, Kovács, Kutnar, Marušič)

What is the intersection density of $\mathrm{Sym}(n)$ with its action on $\mathrm{Sym}(n)/\mathbb{Z}_3$?

1. Find the largest set S of permutations in $\mathrm{Sym}(n)$ so that for any $x, y \in S$
$$xy^{-1} \text{is a 3-cycle.}$$
2. Assume identity is in S; so all other elements are 3-cycles.
3. Consider the elements
$$(1,2,3),\ (1,2,4),\ \ldots,\ (1,2,n)$$
4. This set has size $1 + (n-2) = n-1$.
5. No larger set is possible, since any cycle that intersects with $(1,2,3)$ must be of the form
$$\{(1,2,x), \quad (1,x,3), \quad (x,2,3)\}.$$
6. The intersection density is
$$|S|\left(\frac{|G|}{n}\right)^{-1} = (n-1)\left(\frac{n!}{\frac{n!}{3}}\right)^{-1} = \frac{n-1}{3}.$$

First Tool : Clique-Coclique Bound

Lemma

Clique-coclique bound for a vertex-transitive graph X

$$\omega(X)\,\alpha(X) \leq |V(X)|.$$

Proposition

If a transitive permutation group has a sharply transitive set, then its intersection density is exactly 1.

Proof. 2

A sharply transitive set H has size the degree, n, and no two elements of H are intersecting. So H is a clique of size n in the derangement graph.
By clique/coclique bound, an intersecting set is no larger than $|G|/n$.

Corollary (Frankl and Deza, 1977)

The symmetric group $\mathrm{Sym}(n)$ *has intersection density 1.*

Complete Multi-partite Derangement graphs

Theorem (Meagher, Razafimahatratra, Spiga)

Let G be a transitive permutation group with degree n. If $n \geq 3$, then the derangement graph of G contains a triangle. Moreover, any group G with degree $n \geq 3$, has intersection density $\frac{n}{3}$.

Theorem (Meagher, Razafimahatratra, Spiga)

The derangement graph for a transitive group with degree n is not bipartite if $n > 2$.

Proof. 3

If Γ_G bipartite, then one part is a normal subgroup H that fixes the bipartition. The subgroup H has no derangements, so it has 2 orbits in the group action. If ω and ω' are from different orbits, we can show

$$H = \bigcup_{h \in H} H_{\omega}^{h} \cup \bigcup_{h \in H} H_{\omega'}^{h}.$$

This means that H has **normal covering number two**. By examining the characterization of groups with normal cover two (by M. Garonzi, A. Lucchini), such H exists only if $n = 2$.

- The derangement graph is the **Cayley graph**

$$\mathrm{Cay}(G, \mathrm{der}(G))$$

where $\mathrm{der}(G)$ is the set of derangements of G.
The vertices are elements of G and x, y are adjacent if $xy^{-1} \in \mathrm{der}(G)$.

- The set $\mathrm{der}(G)$ is closed under conjugation; the derangement graph is a **normal Cayley graph**
- If $Cay(G, C)$ is a normal Cayley graph, then the eigenvalues of the adjacency matrix are

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{\sigma \in C} \chi(\sigma)$$

where χ is an irreducible character of G.

- For χ an irreducible character of G, the eigenvalue of Γ_G belonging to χ is

$$\lambda_\chi = \sum_{\substack{C \text{a conjugacy class of derangements} \\ c \in C}} \frac{|C|\, \chi(c)}{\chi(1)}$$

Ratio Bound

Delsarte-Hoffman Ratio Bound

If X is a d-regular graph then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where τ is the least eigenvalue of the adjacency matrix of X.

There is a weighted version of this twoo!

Character Table of $\mathrm{Sym}(5)$

Example

Consider $\mathrm{Sym}(5)$ and the action is on the 2-sets.

()	(1,2)	(1,2)(3,4)	(1,2,3)	(1,2,3)(4,5)	(1,2,3,4)	(1,2,3,4,5)
1	-10	15	20	-20	-30	24
1	-5	0	5	5	0	-6
1	-2	3	-4	-4	6	0
1	0	-5	0	0	0	4
1	2	3	-4	4	-6	0
1	5	0	5	-5	0	-6
1	10	15	20	20	30	24

The eigenvalues of the the derangement graph are: $54, -6, -6, 4, 6, -6, -6$
By Delsarte-Hoffman ratio bound

$$\alpha \leq \frac{120}{1 - \frac{54}{-6}} = 12.$$

The size of a stabilizer of a point under this action is $\frac{|\mathrm{Sym}(5)|}{\binom{5}{2}} = 12.$

Definition

Let $n \geq 2$. Define
$$\mathcal{I}_n := \{\rho(G) \mid G \text{ transitive of degree } n\}\,.$$

The set of all possible intersection densities for a transitive group on an n-set.

The set $\mathcal{I}_n$ is a finite set of rational numbers, so we define
$$I(n) = \max\{\mathcal{I}_n\}.$$

General problems:

1. For a given n, can we determine $\mathcal{I}_n$?
2. For a given n, can we determine $I(n)$?
3. If $I(n)$ is larger than 1, can we determine the structure of the transitive groups G of degree n with $\rho(G) = I(n)$?

Lemma

If G is transitive of prime degree n, then $\rho(G) = 1$ and $\mathcal{I}_n = \{1\}$.

Proof. 4

Let G be transitive of degree n, with n a prime number, and let P be a Sylow n-subgroup of G. Then P is a regular group and hence it is a clique of size n in Γ_G. Thus, from the clique-coclique bound, we have $\rho(G) = 1$ and $\mathcal{I}_n = \{1\}$.

Theorem (Hujdurović, Kovács, Kutnar, Marušič)

If G is a transitive group with degree pq for p and q odd primes, then the intersection density is of G either 1 or 2.

They characterize all such groups with intersection density 2.

Other Problems

Other researchers have suggested other EKR properties:

- Li, Song, Pantagi: Consider intersecting **groups**.
- Bardestani and Mallahi-Karai:
  Consider the intersection density over **all actions** of the group. So consider the action of G on G/H for every subgroup H.

Other questions:

- Which transitive groups have "interesting" intersecting sets of permutations? (We have samples where the maximum set is a subgroup or the union of subgroups. What else can happen?)
- Consider permutation groups that are the automorphism group of a graph.
- What graphs can be derangement graphs?
- Razafimahatratra - gives two new families of transitive groups with complete multipartite derangement graphs. What other complete multipartite graphs can be derangement graphs?