Erdős-Ko-Rado Theorems for Permutations

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Outline

Outline for section 1

The Problem

For a permutation group $G \leq \mathrm{Sym}(n)$, what is the largest set of permutations $\mathcal{F} \subseteq G$ so that for any $\pi, \sigma \in \mathcal{F}$ there is at least one $i \in \{1, \ldots, n\}$ so that

$$\pi(i) = \sigma(i)?$$

Example

In $\mathrm{Sym}(4)$

$$(1,2,3), \quad (2,3,4), \quad (1,2)(3,4), \quad (1,2,3,4)$$

Definition

Permutations π and σ are *intersecting* if $\pi(i) = \sigma(i)$ for some i: Equivalently

- $\sigma^{-1}\pi(i) = i$, so i is a **fixed point** for $\sigma^{-1}\pi$.
- $\sigma^{-1}\pi$ is **not a derangement**.

A derangement is a permutation that has no fixed points.

Examples of Intersecting Groups

Lemma

If $H \leq G$ and H has no derangements, then H is an intersecting set in G.

Let $\pi, \sigma \in H$, then $\sigma^{-1}\pi \in H$, since H is a group.
Since H has no derangements, $\sigma^{-1}\pi$ has a fixed point.

Lemma

If G is a group and $\mathcal{F} \subseteq G$ is an intersecting set, then $x\mathcal{F}$ is intersecting for any $x \in G$.

Let $x\pi, x\sigma \in x\mathcal{F}$, for some $\pi, \sigma \in \mathcal{F}$.
Then

$$(x\sigma)^{-1}\, x\pi = \sigma^{-1}x^{-1}\, x\pi = \sigma^{-1}\, \pi$$

is not a derangement, since $\mathcal{F}$ is intersecting.

We can assume the identity is in any intersecting set, and every other element in the set has a fixed point.

Canonical Intersecting Sets

Definition

For any permutation group G, the **canonical intersecting sets** are

$$S_{i,j} = \{\sigma \in G \,|\, \sigma(i) = j\}.$$

(There are at most n^2 canonical intersecting sets.)

For any group G

$$G_i = S_{i,i} \qquad \text{and} \qquad xG_i = S_{i,j} \quad \text{where } j = x(i).$$

The canonical intersecting sets are the stabilizers of a point, and their cosets.

Lemma

In any transitive group G with degree n the a stabilizer of a point, and its cosets, are intersecting sets of size $\frac{|G|}{n}$.

We will only consider **transitive** groups.

Are the canonical intersecting sets the largest intersecting sets in G?

Group Actions

The intersection is not a property of the group, it is a property of the **group action**.

1. Any transitive group action of a group G is equivalent to the action of G on the cosets G/H for some $H \leq G$.
   Finding **all** actions is as hard as finding all subgroups
2. If $\sigma \in G$ fixes a point in its action on G/H, then there is an x with
   $$\sigma(xH) = xH, \qquad \text{which implies} \qquad x^{-1}\sigma x \in H.$$
3. We are looking for a set $\mathcal{F}$ so that for any $\sigma, \pi \in \mathcal{F}$ we have $\sigma^{-1}\pi$ is conjugate to an element of H.

Outline for section 2

Intersection Density

Definition

Let $G \leq \mathrm{Sym}(n)$ be any transitive group. The *intersection density of the group G* is

$$\rho(G) := \max \left\{ \frac{|\mathcal{F}|}{\frac{|G|}{n}} \mid \mathcal{F} \subseteq G \text{ is intersecting} \right\}.$$

This was defined by Li, Song and Pantangi in 2020.

1. This is the ratio between the size of the largest intersecting set in G and the size of a stabilizer of a point in G.
2. The intersection density of any transitive permutation group is at least 1.
3. Groups with intersection density 1 are also said to have the *Erdős-Ko-Rado Property*.
4. The intersection density is greater than 1 if and only if there is an intersecting set larger than the stabilizer of a point.

Questions about Intersection Density

1. How big can the intersection density be?
2. How can we find bounds on the intersection density?

3. What groups have intersection density 1?
4. Can we characterize the groups with intersection density 1?
5. Are there other group properties that imply intersection density 1?

6. The intersection density is clearly rational, when is it an integer?
7. For a given n what intersection densities can the groups in in $\mathrm{Sym}(n)$ have?

An Unusual Action

Example (Hujdurović, Kovács, Kutnar, Marušič)

Let $H = \{(), (1,2,3), (1,3,2)\} \leq \mathrm{Sym}(k)$.
What is the intersection density of $\mathrm{Sym}(k)$ with its action on $\mathrm{Sym}(k)/H$?

1. The degree of this action is $n = k!/3$.
   The degree of the natural action of $\mathrm{Sym}(k)$ is k.
2. Find the largest set $\mathcal{F}$ of permutations in $\mathrm{Sym}(n)$ so that for any $\sigma, \pi \in \mathcal{F}$
   $$\sigma^{-1}\pi \quad \text{is a 3-cycle.}$$
3. Assume identity is in $\mathcal{F}$; all other elements are 3-cycles, assume $(1,2,3) \in \mathcal{F}$.
4. Any cycle that intersects with $(1,2,3)$ must be of the form
   $$\{(1,2,x), (1,x,3), (x,2,3)\}.$$
5. A maximum set is: $\{(), (1,2,3),\ (1,2,4),\ \ldots,\ (1,2,k)\}$.
6. This set has size $1 + (k-2) = k-1$ and is intersecting.
7. The intersection density is $\frac{k-1}{\frac{k!}{k!/3}} = (k-1)/3$.

Outline for section 3

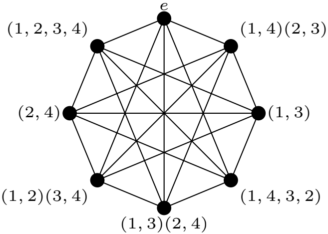
Derangement Graph

Definition

For any $G \leq \mathrm{Sym}(n)$ we can define a **derangement graph**, Γ_G.

- The vertices are the elements of G.
- Vertices $\sigma, \pi \in G$ are adjacent if and only if $\sigma^{-1}\pi$ is a derangement.

(So permutations are adjacent if they are **not** intersecting.)



The graph $\Gamma_{D(4)}$.

Properties of the Derangement Graphs

An intersecting set in G is a **coclique** (independent set) in Γ_G.

$\alpha(\Gamma_G)$ is the size of the largest coclique in the derangement graph of G.

For a transitive group G, what is $\alpha(\Gamma_G)$?

- This graph is **regular**, all the vertices have the same number of neighbours.
- The **degree** is the number of derangements.
- A **semi-regular** subgroup is a clique.
- The derangement graph is the **Cayley graph** $\mathrm{Cay}(G, \mathrm{der}(G))$ where $\mathrm{der}(G)$ is the set of derangements of G.
  The vertices are elements of G, with σ, π are adjacent if $\sigma^{-1}\pi \in \mathrm{der}(G)$.
- G is a subgroup of the automorphism group of Γ_G.
- This graph is **vertex transitive** the automorphism group acts transitively on the vertices (all the vertices are the same).

Connected Derangement graphs

For which groups G is Γ_G connected?

Theorem

$\mathrm{Cay}(G, C)$ *is connected if and only if* C *generates the group, so* $G = \langle C \rangle$.

Example

The derangement graph of any Frobenius group is the disjoint union of complete graphs. If G is a Frobenius group, then $G = K \rtimes H$; where H has no derangements. All elements of K, except the identity, are derangements.

> *"In most cases, $\langle der(G) \rangle = G$. For example, of the 3,302,368 transitive groups of degree from 2 to 47 inclusive as classified in and available in Magma, only 893 have $\langle der(G) \neq G$ (of which 103 are Frobenius groups);"*

from "Groups generated by derangements" -R.A. Bailey, Peter J. Cameron, Michael Giudici, Gordon F. Royle, 2021

Outline for section 4

Graph Homomorphisms

Let X and Y be graphs.
A **graph homomorphism** is a map $\phi : V(X) \to V(Y)$ that maps adjacent vertices in X to adjacent vertices in Y.

Theorem

Let X and Y be vertex-transitive graphs with $X \to Y$, then

$$\frac{|V(X)|}{\alpha(X)} \leq \frac{|V(Y)|}{\alpha(Y)}$$

(this is the fractional chromatic number.)

Minimal Transitive Subgroups

Theorem

Let G, H be transitive groups with degree n and $H \leq G$, then

$$\rho(G) \leq \rho(H)$$

$\rho(G)$ is the intersection density of G.

Since $H \leq G$ embedding is a homomorphism

$$\Gamma_H \to \Gamma_G$$

so

$$\alpha(\Gamma_G) \leq \frac{|G|}{|H|}\alpha(\Gamma_H).$$

Further,

$$\rho(G) = \alpha(\Gamma_G)\left(\frac{|G|}{n}\right)^{-1} \leq \frac{|G|}{|H|}\alpha(\Gamma_H)\frac{n}{|G|} = \alpha(\Gamma_H)\frac{n}{|H|} = \rho(H).$$

We can prove $\rho(G) = 1$, by proving G has a transitive subgroup H with $\rho(H) = 1$.

Clique-Coclique Bound

Lemma (Clique-coclique bound)

Let X be a vertex-transitive graph, then $\omega(X)\,\alpha(X) \leq |V(X)|$.

Embedding is a homomorphism $K_{\omega(X)} \to X$, so

$$\alpha(X) \leq |V(X)| \frac{\alpha(K_{\omega(X)})}{|K_{\omega(X)}|} = \frac{|V(X)|}{\omega(X)}.$$

Lemma

If $H \leq G$ and all non-identity elements of H are derangements then

$$\rho(G) \leq \frac{n}{|H|}.$$

A subgroup H of derangements is a clique of size $|H|$.
By clique/coclique bound, an intersecting set is no larger than $|G|/|H|$, so

$$\rho(G) \leq \frac{|G|}{|H|}\frac{n}{|G|} = \frac{n}{|H|}.$$

Sharply Transitive Subgroups

Lemma

Let $G \leq \mathrm{Sym}(n)$ be a group that has a sharply transitive (regular) subgroup, then the intersection density of G is 1.

A sharply transitive subgroup is a clique of size n. By the previous, an intersecting set cannot be larger than $\frac{|G|}{n}$.
Since G is transitive subgroup, so the size of the stabilizer of a point is $\frac{|G|}{n}$.

Corollary (Deza and Frankl, 1977)

The largest intersecting set of permutations has size exactly $(n-1)!$.

Or, $\mathrm{Sym}(n)$ has intersection density 1.

Lemma

Let $G \leq \mathrm{Sym}(n)$ be a group that has a semi-regular subgroup (only the identity has fixed points) of size k, then the intersection density of G is at most n/k.

An intersecting set is no larger than $\frac{|G|}{k}$, so the density is no larger than $\frac{\frac{|G|}{k}}{\frac{|G|}{n}} = \frac{n}{k}$.

Example of Sharply Transitive Subgroups

Example

Consider $\mathrm{Alt}(4)$ with the natural action on $\{1,2,3,4\}$. The stabilizer of a point has size $12/4 = 3$. The subgroup

$$H = \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

is sharply transitive with size 4 (the degree), with this action the intersection density is 1.

Example

Consider $\mathrm{Alt}(4)$ acting on pairs: $\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$.

The stabilizer of a point, is the subgroup

$$H = \{(), (1,2)(3,4)\}$$

(Since $\sigma(i) = (1,2)(3,4)\ (\{1,2\}) = \{1,2\}$.)

But, the following subgroup is intersecting and twice the size of the stabilizer of a point

$$H = \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

There is a semi-regular subgroup $\{(), (1,2,3), (1,3,2)\}$ of size 3, so

$$\rho(\mathrm{Alt}(4)') \leq \frac{n}{k} = \frac{6}{3} = 2.$$

The intersection density of $\mathrm{Alt}(4)$ with this action is 2.

Outline for section 5

Eigenvalues of Cayley Graphs

1. The derangement graph is a *normal* Cayley graph

$$\Gamma_G = \mathrm{Cay}(G, \mathrm{der}(G)).$$

2. The *connection set* of the Cayley graph is set of derangements; so the connections set is closed under conjugation.

Theorem

$\Gamma_G = Cay(G, \mathrm{der}(G))$ is the derangement graph for a permutation group G, and its eigenvalues are

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{\sigma \in \mathrm{der}(G)} \chi(\sigma)$$

where χ is an irreducible character of G.

Example

Let $\mathbf{1}$ be the trivial character for G, then

$$\lambda_{\mathbf{1}} = \frac{1}{\mathbf{1}(1)} \sum_{g \in \mathrm{der}(G)} \mathbf{1}(g) = |\,\mathrm{der}(G)| = d.$$

This is the degree of the derangement graph.

Delsarte Hoffman's Ratio Bound

Ratio Bound

If X is a d-regular graph then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where d is the degree and τ is the least eigenvalue for the adjacency matrix for X.

If

- equality holds in the ratio bound
- and y is a characteristic vector for a maximum coclique,

then

$$y - \frac{\alpha(X)}{|V(X)|}\mathbf{1}$$

is an eigenvector for τ.
This can be used to characterize all the maximum cocliques in the graph.

Intersecting Permutations in 2-Transitive Groups

Let G be a 2-transitive group

- $\chi(g) = \mathrm{fix}(g) - 1$ is an irreducible character of G,
- its eigenvalue is

$$\tau = -\frac{|Der(G)|}{n-1} = -\frac{d}{n-1}.$$

- Putting this into the ratio bound gives

$$\alpha(\Gamma_G) \leq \frac{|G|}{1 - \frac{d}{-\frac{d}{n-1}}} = \frac{|G|}{n}.$$

So if this eigenvalue is **the least** eigenvalue then the group has the EKR property.

Theorem (Meagher, Spiga, Tiep)

All 2-transitive groups have intersection density 1.

First we used the two reductions:

1. If a group has a sharply transitive subgroup, then the group has intersection density 1.
2. If G has a transitive subgroup H with $\rho(H) = 1$, then $\rho(G) = 1$.

We only needed to look at minimal transitive subgroups of almost simple type.

1. These are classified (shortlist!)
2. Ratio bound held for each family, but some need a **weighing**
3. or some extra work to show we had the **least** eigenvalue.

Outline for section 6

Basic Bounds on Intersection Density

Proposition

If G is a transitive group of degree $n \geq 2$, then $\rho(G) \leq n/2$.

By Jordan's theorem G has a derangement, so Γ_G has at least one edge. The clique-coclique bound implies that $\alpha(\Gamma_G) \leq |G|/2$.

Can this bound be reached?

Example

The group $\mathrm{Sym}(2)$ with natural action on $\{1, 2\}$ has $\rho(\mathrm{Sym}(2)) = 1 = \frac{2}{2}$

Note that $\rho(G) = n/2$ implies there is an intersecting set $\mathcal{F}$

$$\frac{|\mathcal{F}|}{\frac{|G|}{n}} = \frac{n}{2} \qquad \text{if} \qquad |\mathcal{F}| = \frac{|G|}{2}$$

Can Γ_G be bipartite? We checked groups using GAP.

Bipartite Derangement Graph

Theorem (Meagher, Razafimahatratra, Spiga)

The derangement graph for a transitive group with degree n is not bipartite if $n > 2$.

$\mathrm{Sym}(2)$ is the only group with a bipartite derangement graph.

Theorem (Meagher, Razafimahatratra, Spiga)

Let $G \leq \mathrm{Sym}(n)$ be a transitive permutation group. If $n \geq 3$, then the derangement graph of G contains a triangle.

Using the clique-coclique bound, this result leads to the following corollary.

Corollary

For any group G with degree $n \geq 3$, we have $\rho(G) \leq \frac{n}{3}$.

Question

Are there lots of groups with $\rho(G) = \frac{n}{3}$?

Example of a Derangement Graph

Example (Razafimahatratra)

Let `G`:=TransitiveGroup(18, 142).

1. This is a transitive group with size 324.
2. It is imprimitive (has a system with three blocks of size six and another with six blocks of size three.)
3. The eigenvalues of the derangement graph for this group are
$$\{216, 0, -108\}$$
4. This means that the derangement graph for this graph is a complete tripartite graph.
5. $$\rho(G) = 108\frac{18}{324} = 6 = \frac{n}{3}.$$

We only found four groups searching with Gap, but could not find a construction!

Multipartite Derangement Graphs

1. When is the derangement graph a complete multi-partite graph?
2. Can any multi-partite graph be a derangement graph?
3. Do we get the maximum intersection density with groups whose derangement graph is complete multipartite?

Observation

The derangement graph for any degree n group is an n-partite graph.

Let G be a group acting on the set $\{1, 2, \dots, n\}$, then the sets

$$S_{1,1},\ S_{1,2},\ S_{1,3},\ \dots,\ S_{1,n}$$

form a partition of the vertices. There are no edges within an $S_{1,i}$.

When is the derangement graph for a degree n graph a k-partite graph for $k < n$?

Chromatic Number of a Derangement Graph

For any $G \leq \mathrm{Sym}(n)$, the chromatic number of the derangement graph is bounded $\chi(\Gamma_G) \leq n$.

An n-colouring exists with the colour classes:

$$S_{1,1},\ S_{1,2},\ S_{1,3},\ \ldots,\ S_{1,n}.$$

Lemma

If a group $G \leq \mathrm{Sym}(n)$ has intersection density 1, then $\chi(\Gamma_G) = n$.

Since the size of a colour class is no larger than $\alpha(\Gamma_G)$,

$$\chi(\Gamma_G) \leq \frac{|G|}{\alpha(\Gamma_G)} = n.$$

For which groups G is $\chi(\Gamma_G) < n$?

Outline for section 7

The EKR-Type Properties

What are the largest intersecting sets of permutations in a group $G \leq \mathrm{Sym}(n)$.

A group $G \leq \mathrm{Sym}(n)$ has the:

1. **EKR property** if the maximum coclique in Γ_G has *size* $\frac{|G|}{n}$.
(Canonical intersecting sets have the largest size.)

2. **EKR-module property** if the characteristic vector of any maximum coclique is contained in the vector space

$$V = span\{v_{i,j} \,|\, i, j \in [n]\},$$

where $v_{i,j}$ is the characteristic vector of the permutations in G that map i to j.
(This is a *weak characterization*)

3. **strict-EKR property** if the maximum cocliques in Γ_G are the sets $S_{i,j}$ (the canonical cocliques).
(Canonical intersecting sets are the only intersecting sets with the largest size.)
(This is a *strong characterization*)

Cameron-Leibler Sets for Permutations

Theorem (M. and Sin)

All 2-transitive groups have EKR module property.

If G is 2-transitive,

1. The characteristic vector of any maximum coclique is in V.
2. To prove G has the **strict-EKR property** we can show that the only 01-vectors in V with weight $|G|/n$ are the sets $v_{i,j}$.

Define a **CL-set** in G to be a set $S \subseteq G$ with characteristic vector of S in V.

For a group G what are all the CL-sets?

Theorem (Ellis, 2011)

Consider the natural action of $\mathrm{Sym}(n)$ *on* $[n]$*. The only CL sets are the canonical CL sets.*

Outline for section 8

Variations of EKR for Permutations

- Li, Song, Pantagi: Considered characterizing intersecting **groups**.
- Bardestani and Mallahi-Karai: Considered groups that have intersection density 1 for every group action.

- What are all the intersection densities of groups with degree n?
  If G is a transitive subgroup of $\mathrm{Sym}(p)$, where p is a prime, then $\rho(G) = 1$.

- What graphs can be derangement graphs?
- Razafimahatratra - gives two new families of transitive groups with complete multipartite derangement graphs. What other complete multipartite graphs can be derangement graphs?

- David Ellis, Nathan Keller, and Noam Lifshitz (and others) consider stability result

  "Stability versions of such theorems assert that if the size of a family is close to the maximum possible size, then the family itself must be close (in some appropriate sense) to a maximum-sized family. "

EKR Theorems for Other Objects

Intersecting Trees

- Start all trees on the same set of n vertices.
- Two trees Intersect if they have a common edge.

What is the largest family of intersecting trees?

Intersecting Triangulations

- Start with a convex n-gon.
- Make a triangulation by adding $n-3$ edges that only intersect at vertices of the n-gon.
- Intersect if they have a common triangle.

What is the largest family of intersecting triangulations?

MathOverflow. mathoverflow.net/q/114646.