Intersecting Sets of Uniform Partitions

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Partitions

Definition

A \((k, \ell)\)-partition is a set partition of \(\{1, 2, \ldots, k\ell\}\) with exactly \(\ell\) blocks each of size \(k\).

\[
P = \{P_1, P_2, \ldots, P_\ell\}
\]

These are also called uniform set partitions. The set of all uniform-\((k, \ell)\) partitions is denoted by \(\mathcal{U}(k, \ell)\).

Two set partitions

\[
P = \{P_1, P_2, \ldots, P_\ell\} \quad \text{and} \quad Q = \{Q_1, Q_2, \ldots, Q_\ell\}
\]

1. are intersecting if \(P_i = Q_j\) for some \(i\) and \(j\).
2. are \(t\)-intersecting if \(P_{i_1} = Q_{j_1}, P_{i_2} = Q_{j_2}, \ldots, P_{i_t} = Q_{j_t}\) for distinct \(i_1, \ldots, i_t\) and distinct \(j_1, \ldots, j_t\).
3. are partially-\(t\) intersecting if \(|P_i \cap Q_j| \geq t\) for some \(i\) and \(j\).
Intersecting sets

Example

\[ P1 : \quad 1 \ 2 \ 3 \ | \ 4 \ 5 \ 6 \ | \ 7 \ 8 \ 9 \ | \ 10 \ 11 \ 12 \]

\[ P2 : \quad 1 \ 2 \ 3 \ | \ 4 \ 5 \ 6 \ | \ 7 \ 8 \ 10 \ | \ 9 \ 11 \ 12 \]

\[ P3 : \quad 1 \ 4 \ 7 \ | \ 2 \ 5 \ 10 \ | \ 3 \ 8 \ 11 \ | \ 6 \ 9 \ 12 \]

Definition

A set of partitions is (intersecting / \( t \)-intersecting / \( t \)-partially intersecting) if the partitions are pairwise (intersecting / \( t \)-intersecting / \( t \)-partially intersecting).

What is a maximum set of (intersecting / \( t \)-intersecting / \( t \)-partially intersecting) partitions?

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Previous Results for \( t \)-intersection

The total number of \((k, \ell)\)-partitions is

\[
U(k, \ell) = \frac{1}{\ell!} \binom{k\ell}{k} \binom{k\ell - k}{k} \cdots \binom{k}{k}.
\]

Definition

A canonical \( t \)-intersecting set is all partitions containing a fixed set of \( t \) disjoint parts.

Example (1-intersecting \((3, 3)\)-partitions)

\[
\begin{array}{ccc}
123 | 456 | 789 & 123 | 457 | 689 & 123 | 458 | 689 \\
123 | 459 | 678 & 123 | 467 | 589 & 123 | 468 | 579 \\
123 | 469 | 578 & 123 | 478 | 569 & 123 | 479 | 568 \\
123 | 489 | 567
\end{array}
\]

The size of a canonical \( t \)-intersecting \((k, \ell)\)-partition is

\[
U(k, \ell - t) = \frac{1}{(\ell - t)!} \binom{k\ell - kt}{k} \binom{k\ell - k(t + 1)}{k} \cdots \binom{k}{k}.
\]
If $k = 2$, the partitions are perfect matchings.

**Theorem**

The largest $t$-intersecting set of perfect matchings are the canonical sets in the following cases:

- for $t = 1$ (Godsil and Meagher, 2017)
- for $t = 2$ (Fallat, Meagher and Shirazi, 2021; Lindzey, 2023).
- for all $t$ and $n$ sufficiently large (Lindzey, 2017)

Proofs are algebraic, using eigenvalues of a related graph.

**Proposition**

If $n$ is small relative to $t$ there are examples of $t$-intersecting sets larger than the canonical sets.

**Conjecture**

If $k \geq 3t/2 + 1$ then the largest set of $t$-intersecting perfect matchings are the canonical.
Trivial Cases for Partially Intersection

1. Any two \((k, \ell)\)-uniform partially 1-intersecting.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
\end{array}
\begin{array}{cccccc}
1 & 4 & 7 & 3 & 5 & 10 \\
2 & 8 & 11 & 6 & 9 & 12 \\
\end{array}
\]

2. If \(t = k\), then partially-\(t\) intersecting and intersecting are the same.

3. If \(k > \ell(t - 1)\), then any two \((k, \ell)\)-partitions are partially \(t\)-intersecting.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
\end{array}
\begin{array}{cccccc}
1 & 2 & 3 & 7 & 8 & 9 \\
4 & 5 & 6 & 10 & 11 & 12 \\
\end{array}
\]

if the parts are really big, any two partitions will be partially \(t\)-intersecting.

Focus on \((k, \ell)\)-uniform partially 2-intersecting sets with \(k > 2\).

\(P = \{P_1, P_2, \ldots, P_\ell\}\) and \(Q = \{Q_1, Q_2, \ldots, Q_\ell\}\) have some \(i, j\) with \(|P_i \cap Q_j| \geq 2\).
Definition

For \( t \leq k \), fix a \( t \)-subset \( T \subset \{1, \ldots, k\ell\} \). The set of all partitions that have a part containing \( T \) is a canonical \( t \)-intersecting set of partitions.

Example

Canonical partially \( 2 \)-intersecting set of \((3, 3)\)-partitions:

\[
\begin{array}{cccc}
123|456|789 & 123|457|689 & 123|458|679 & 123|459|678 \\
123|467|589 & 123|468|579 & 123|469|578 & 123|567|489 \\
\ldots & \ldots & \ldots & \ldots \\
129|356|478 & 129|367|458 & 129|368|457 & 129|378|456 \\
\end{array}
\]

The size of a canonical partially \( t \)-intersecting set of partitions is

\[
\frac{1}{(\ell - 1)!} \left( \begin{array}{c} k\ell - t \\ k - t \end{array} \right) \left( \begin{array}{c} k\ell - k \\ k \end{array} \right) \cdots \left( \begin{array}{c} k \\ k \end{array} \right).
\]
Conjecture

If $k \leq \ell(t - 1)$, then the largest set of intersecting $(k, \ell)$-partitions is a canonical set of $t$-intersecting partitions.
**Derangement Graph**

**Definition**

For any \( k, \ell \) define the **partition derangement graph**, \( \Gamma_{G(k, \ell)} \).

- The vertices are the uniform \((k, \ell)\)-partitions.
- Vertices \( P, Q \in \mathcal{U}(k, \ell) \) are adjacent if and only if \( P \) and \( Q \) are **not** partially \( t \)-intersecting.

A set of uniform partitions is a coclique (independent set) in \( \Gamma_{G(k, \ell)} \), exactly if the set is a partially \( t \)-intersecting set of partitions.

**Graph Properties:**

1. The graph \( \Gamma_{G(k, \ell)} \) is regular, denote the degree by \( d_{k, \ell} \).
2. The derangement graph is vertex-transitive, the group \( \text{Sym}(k\ell) \) acts transitively on the vertices of \( \Gamma_{G(k, \ell)} \).
3. A resolvable balanced incomplete block \( t \)-design on \( k\ell \) points with blocksize \( k \) and index \( \lambda = 1 \), if it exists, is a maximum clique.
Resolvable Designs

**Definition**

Suppose $\mathcal{B}$ is a $t-(n, k, \lambda)$ design.

- A *parallel class* in $\mathcal{B}$ is a collection of disjoint sets whose union is the $n$-set.

- A partition of $\mathcal{B}$ into $r = \lambda \frac{(n-1)}{(k-1)(t-1)}$ parallel classes is called a *resolution*.

- A $t-(n, k, \lambda)$ design is *resolvable* if a resolution exists.

**Example (Resolvable $2-(9, 3, 1)$ Design)**

123 | 456 | 789 (orange)

147 | 258 | 369 (red)

159 | 267 | 348 (green)

168 | 249 | 357 (blue)
Clique/Coclique Bound

Theorem

If $X$ is a vertex-transitive graph and $\alpha(X)$ is the size of the maximum coclique and $\omega(X)$ the size of the maximum clique, then

$$\alpha(X) \omega(X) = |V(X)|.$$ 

Theorem

If there is a resolvable balanced incomplete block $t$-design on $k\ell$ points with blocksize $k$, then a canonical partially $t$-intersecting partitions is a largest intersecting set.

Proof. By the clique/clique bound, a coclique is no larger than

$$\frac{\text{number vertices}}{\text{clique size}} = \frac{U(k, \ell)}{\frac{1}{\ell} \binom{k\ell}{t} \binom{k}{t}} = \frac{1}{(\ell - 1)!} \binom{k\ell - t}{k - t} \binom{k\ell - k}{k} \binom{k\ell - 2k}{k} \cdots \binom{k}{k}$$

= the size of a canonical partially $t$-intersecting set
Theorem (Delsarte-Hoffman bound)

Let $A$ be the adjacency matrix for a $d$-regular graph $X$ on vertex set $V(X)$. If the least eigenvalue of $A$ is $\tau$, then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}.$$ 

If equality holds for some coclique $S$ with characteristic vector $\nu_S$, then

$$\nu_S - \frac{|S|}{|V(X)|} \mathbf{1}$$

is an eigenvector with eigenvalue $\tau$.

Master Plan:

1. find 3 specific eigenvalues of the graph,
2. show all other eigenvalues are smaller, (in absolute value)
3. apply the ratio bound.
Eigenspaces of the Derangement graphs

1. The eigenspaces of $X_{k,\ell}$ are invariant under the action of $\text{Sym}(k\ell)$ and thus a union of irreducible modules in the decomposition of

$$\text{ind} \left( 1_{\text{Sym}(k)\wr\text{Sym}(\ell)} \right)^{\text{Sym}(k\ell)}.$$ 

Every irreducible representation of $\text{Sym}(n)$ is labelled by an integer partition of $n$.

2. For each irreducible representation in $\text{ind} \left( 1_{\text{Sym}(k)\wr\text{Sym}(\ell)} \right)^{\text{Sym}(k\ell)}$ has a corresponding eigenvalue:

$$\eta_\phi(A_\ell) = \frac{d_\ell}{|H|} \sum_{x_\ell} \sum_{h \in H} \phi(x_\ell h),$$

This includes a sum of characters over group cosets which is hard.

3. The irreducible representations $[k\ell], [k\ell - 2, 2]$ and $[k\ell - 3, 3]$ are included in $\text{ind} \left( 1_{\text{Sym}(k)\wr\text{Sym}(\ell)} \right)^{\text{Sym}(k\ell)}$.

4. The irreducible representations $[1^{k\ell}], [k\ell - 1, 1], [2, 1^{k\ell-2}], [2, 2, 1^{k\ell-4}], [k\ell - 2, 1, 1], [3, 1^{k\ell-3}], [2, 2, 2, 1^{k\ell-6}]$ are not.

Proof by orbit counting.
Eigenvalues of the Derangement graphs

**Theorem**

For any \( k, \ell \),

1. **The eigenvalue belonging to** \([k \ell]\) **is the degree** \( d = d(k, \ell) \).

2. **The eigenvalue belonging to** \([k \ell - 2, 2]\) **is**
   
   \[ \tau = -\frac{(k - 1)d}{k(\ell - 1)}. \]

3. **The eigenvalue belonging to** \([k \ell - 3, 3]\) **is**
   
   \[ \theta = \frac{2(k - 1)(k - 2)d}{k^2(\ell - 1)(\ell - 2)}. \]

4. **For any other representation, the eigenvalue is smaller in absolute value** \( \tau \).

By the ratio bound, the maximum size of coclique in \( X_{k, \ell} \) is

\[
\frac{|V(X_{k, \ell})|}{1 - \frac{d}{\tau}} = \frac{v}{1 - \frac{d}{\frac{(k-1)d}{k(\ell-1)}}} = \frac{v}{1 + \frac{k(\ell-1)d}{k-1}} = \frac{v(k-1)}{k\ell - 1} = \binom{k\ell - 2}{k-2} U(k, \ell - 1).
\]
The action of a subgroup of the automorphism group on the partitions forms orbits. These orbits can be used to build a **quotient graph**.

Partition the vertices in the graph into orbits.

\[
\begin{pmatrix}
A & B & C \\
A & (a & b & c) \\
B & (d & e & f) \\
C & (g & h & i)
\end{pmatrix}
\]

The **quotient matrix**
Quotient Graphs

Young’s subgroup $\text{Sym}(k\ell)$:
- Has one orbit, so the quotient graph is $(d)$.
- This means $d$ is the eigenvalue corresponding to $[k\ell]$.

Young’s subgroup $\text{Sym}([k\ell - 2, 2]) = \text{Sym}(k\ell - 2) \times \text{Sym}(2)$
- Has two orbits: the partitions with 1 and 2 together in one part and the partitions where they are in two parts.
- The quotient matrix is the $2 \times 2$ matrix
  \[
  \begin{pmatrix}
  0 & d \\
  -\tau & d + \tau
  \end{pmatrix}
  \]
- The eigenvalues are $d$ and $\tau = -\frac{d(k-1)}{k(\ell-1)}$.
- This means $-\tau$ is the eigenvalue corresponding to $[k\ell - 2, 2]$.
Young’s subgroup $\text{Sym}([k\ell - 3, 3]) = \text{Sym}(k\ell - 3) \times \text{Sym}(3)$

- For $[k\ell - 3, 3]$, the Young’s subgroup is $\text{Sym}(k\ell - 3, 3)$ has three orbits: partitions where 1,2,3 are in one part, two parts or three parts.

- The quotient matrix is the $3 \times 3$ matrix

$$M = \begin{pmatrix} 0 & 0 & d \\ 0 & a & d - a \\ b & c & d - b - c \end{pmatrix}$$

- The eigenvalues are $d, -\tau, \theta$.

- $\text{tr}(M) = d - b - c + a = d - \tau + \theta$

- Counting edges between the orbits gives equations for $a, b, c$, then

$$\theta = \frac{2(k - 1)(k - 2)d}{k^2(\ell - 1)(\ell - 2)}.$$
Assume $k\ell \geq 13$ and $k \geq 3$. Then the only partitions in the decomposition of $\text{ind} \left( 1_{\text{Sym}(k) \wr \text{Sym}(\ell)} \right)^{\text{Sym}(k\ell)}$ with dimension less than or equal to $\binom{k\ell}{3} - \binom{k}{2}$ are

\[ \chi_{[k\ell]}, \quad \chi_{[k\ell-2,2]}, \quad \chi_{[k\ell-3,3]} \cdot \]

**Proof.** Use induction and the "branching rule". \qed
Theorem

The eigenvalues $d$, $\tau$ and $\theta$ are the three largest, in absolute value, in the derangement graph. The smallest eigenvalue for $\Gamma_{(k,\ell)}$ is $\tau$.

Proof. By squaring the adjacency matrix and taking the trace, we have

$$vd = d^2 + m\tau^2 + m\theta^2 + \sum_{i=2}^{j} m_i \lambda_i^2.$$ 

Hence for every $2 \leq i \leq j$ we have

$$vd - d^2 - m\tau^2 - m\theta^2 \geq m_i \lambda_i^2.$$ 

This gives an upper bound on $|\lambda_i|$ in terms of $d$. 

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Theorem (Bender)

Let $M_{k,\ell}$ be the number of all $\ell \times \ell$ matrices with entries either 0 or 1, and row and columns sums equal to $k$. For positive integers $k, \ell$

$$\lim_{\ell \to \infty} \frac{(k!)^{2\ell}}{(k\ell)!} |M_{k,\ell}| = e^{-(k-1)^2/2}.$$ 

Lemma

For positive integers $k, \ell$ with $k \leq \ell$, let $d$ be the degree of $\Gamma_{(k,\ell)}$. Then

$$d = \frac{k!\ell}{\ell!} |M_{k,\ell}|.$$ 

Further, for a fixed integer $k$ with $k \geq 2$,

$$\lim_{\ell \to \infty} \frac{U(k, \ell)}{d} = e^{(k-1)^2/2}.$$ 

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Bound on Degree

We had
\[
\left( v d - d^2 - m_\tau \left( \frac{d(k-1)}{k(\ell-1)} \right)^2 - m_\theta \left( \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)} \right)^2 \right) \frac{1}{m_i} \geq |\lambda_i| 
\]

1. \( \lim_{\ell \to \infty} \frac{U(k,\ell)}{d} = e^{\frac{(k-1)^2}{2}} \) gives an upper bound for \( \lambda_i \).
2. If another eigenvalue is larger than \( \tau \), in absolute value, that eigenvalue has to have a multiplicity smaller than \( \binom{k\ell}{3} - \binom{k\ell}{2} \),
3. Only \( d, \tau \) or \( \theta \) have multiplicity this small.
4. So \( \tau \) is the least eigenvalue of \( \Gamma_{k,\ell} \).
5. Apply the ratio bound.
### Theorem

Fix an integer $k \geq 3$. For $\ell$ sufficiently large, the largest set of partially 2-intersecting uniform $(k, \ell)$-partitions has size $\left(\frac{k\ell - 2}{k-2}\right)U_{k,\ell-1}$.

### Conjecture

For $k \geq 3$ and $\ell$ sufficiently large, the only sets of partially 2-intersecting $(k, \ell)$-partitions with size $\left(\frac{k\ell - 2}{k-2}\right)U_{k,\ell-1}$ are the sets $S_{i,j}$.

With a more tedious calculation on the degree approximation we get the follow:

### Theorem

For $k = 3$ and all $\ell \geq 3$ the largest set of partially 2-intersecting uniform partitions has size

$$ (3\ell - 2)U_{3,\ell-1}. $$
Future Work

There are two obvious questions:

**Question**
Can a non-canonical 2-partially intersecting set also have the maximum size?

**Question**
Can this method be extended $t$-partially intersecting uniform partitions for large values of $t$?