Intersection theorems for Uniform Partitions

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Slides available at: https://uregina.ca/~meagherk/Australia.pdf
The Motivating Problem

What is the largest collection of subsets from \( \{1, 2, \ldots, n\} \), so that any two subsets contain a common element?

Restrict to \( k \)-subsets, subsets with exactly \( k \) elements

Example

This is an example of 3-subsets from \( \{1, 2, \ldots, 6\} \) with size 10:

\[
\begin{align*}
\{1, 2, 3\} & \quad \{1, 2, 4\} & \quad \{1, 2, 5\} & \quad \{1, 2, 6\} & \quad \{1, 3, 4\} \\
\{1, 3, 5\} & \quad \{1, 3, 6\} & \quad \{2, 3, 4\} & \quad \{2, 3, 5\} & \quad \{2, 3, 6\}
\end{align*}
\]

All sets contain at least two elements of \( \{1, 2, 3\} \).

This is an example of 3-subsets from \( \{1, 2, \ldots, 7\} \) with size 15:

\[
\begin{align*}
\{1, 2, 3\} & \quad \{1, 2, 4\} & \quad \{1, 2, 5\} & \quad \{1, 2, 6\} & \quad \{1, 2, 7\} \\
\{1, 3, 4\} & \quad \{1, 3, 5\} & \quad \{1, 3, 6\} & \quad \{1, 3, 7\} & \quad \{1, 4, 5\} \\
\{1, 4, 6\} & \quad \{1, 4, 7\} & \quad \{1, 5, 6\} & \quad \{1, 5, 7\} & \quad \{1, 6, 7\}
\end{align*}
\]

All sets contain 1.
An **intersecting** $k$-**set system** is a collection of subsets of $[1..n]$, each of size $k$, so that any two have at least one element in common.

**Theorem (Erdős-Ko-Rado Theorem - 1961)**

Let $\mathcal{A}$ be an intersecting $k$-set system on an $n$-set. If $n \geq 2k$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

1. A largest collection of intersecting $k$-sets is one in which all the sets that contain a common point.
2. These collections are called **trivially intersecting** or **canonically** intersecting.
Generalizations Erdős-Ko-Rado Theorem

For any object that has an "intersection" we can ask:

What is the size of largest set of intersecting objects?

Which intersecting set attain the maximum size?

In general:

- Each **object** is made of *k* **atoms**.
- Two objects **intersect** if they contain a common atom.
- A **canonically intersecting system** is the set of all objects that contain a fixed atom.

<table>
<thead>
<tr>
<th>Object</th>
<th>Atoms</th>
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<td><em>k</em>-Subsets of [1..n]</td>
<td>elements from {1, \ldots, n}</td>
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<tr>
<td>Integer sequences</td>
<td>pairs ((i, a)) (entry (a) is in position (i))</td>
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<td>pairs ((i, j)) (the permutation maps (i) to (j))</td>
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<td>Perfect matchings in a graph</td>
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**Definition**

A **uniform** \((k, \ell)\)-partition is a set partition of \(\{1, 2, \ldots, k\ell\}\) with exactly \(\ell\) blocks each of size \(k\).

\[
P = \{P_1, P_2, \ldots, P_\ell\}
\]

These are also called **uniform set partitions**.

The number of uniform \((k, \ell)\)-partitions is

\[
U(k, \ell) = \frac{1}{\ell!} \binom{k\ell}{k} \binom{k(\ell - 1)}{k} \cdots \binom{k}{k}
\]

**Example**

A uniform \((3, 4)\)-partition:

\[
1 \ 4 \ 7 \ | \ 2 \ 5 \ 10 \ | \ 3 \ 8 \ 10 \ | \ 6 \ 9 \ 12
\]
Intersecting sets

Two set partitions

\[ P = \{P_1, P_2, \ldots, P_\ell\} \quad \text{and} \quad Q = \{Q_1, Q_2, \ldots, Q_\ell\} \]

1. are **intersecting** if \( P_i = Q_j \) for some \( i \) and \( j \); (contain a common part)

2. are **\( t \)-intersecting** if

\[
P_{i_1} = Q_{j_1}, \quad P_{i_2} = Q_{j_2}, \quad \ldots \quad P_{i_t} = Q_{j_t}
\]

for distinct \( i_1, \ldots, i_t \) and distinct \( j_1, \ldots, j_t \); (contain \( t \) common parts)

1-intersecting is intersecting.

3. are **partially-\( t \) intersecting** if \( |P_i \cap Q_j| \geq t \) for some \( i \) and \( j \). (a pair of parts contain \( t \) common elements).

---

Example

\[
\begin{array}{cccc|cccc|cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 4 & 7 & 2 & 5 & 10 & 3 & 8 & 11 & 6 & 9 & 12
\end{array}
\]
Intersecting Partitions

**Definition**
A set of partitions is \textit{t-intersecting} if the partitions are \textit{pairwise t-intersecting}.

What is a maximum set of \textit{t}-intersecting partitions?

**Definition**
A set of partitions is \textit{t-partially intersecting}, if the partitions are \textit{pairwise t-partially intersecting}.

What is a maximum set of \textit{t}-partially intersecting partitions?
Canonical Intersecting Sets of Partitions

**Definition**
Fix $t$ disjoint $k$-subsets $T_1, T_2, \ldots, T_t$. The set of all partitions that have $T_1, T_2, \ldots, T_t$ as parts is a **canonical $t$-intersecting set of partitions**.

**Example**
A canonical partially 1-intersecting set of $(3, 3)$-partitions with $T = \{1, 2, 3\}$ and size 10:

```
123|456|789  123|457|689  123|458|679  123|459|678  123|467|589
123|468|579  123|469|578  123|478|569  123|479|568  123|489|567
```

The size of a canonical $t$-intersecting set of $(k, \ell)$-partitions is

$$
\frac{1}{(\ell - t)!} \binom{k(\ell - t)}{k} \binom{k(\ell - t - 1)}{k} \cdots \binom{k}{k} = U(k, \ell - t)
$$
Canonical $t$-partially Intersecting Sets of Partitions

**Definition**

For $t \leq k$, fix a $t$-subset $T \subset \{1, \ldots, k\ell\}$.
The set of all partitions that have a part containing $T$ is a **canonical partially $t$-intersecting set of partitions**.

**Example**

A canonical partially 2-intersecting set of (3, 3)-partitions with $T = \{1, 2\}$ and size 70:

| 123|456|789 | 123|457|689 | 123|458|679 | 123|459|678 |
|----|----|----|----|----|----|----|----|----|----|
| 123|467|589 | 123|468|579 | 123|469|578 | 123|567|489 |
| ... |
| 129|356|478 | 129|367|458 | 129|368|457 | 129|378|456 |

The size of a canonical $t$-intersecting set of $(k, \ell)$-partitions is

\[
\binom{k\ell - t}{k - t} \frac{1}{(\ell - 1)!} \binom{k(\ell - 1)}{k} \cdots \binom{k}{k} = \binom{k\ell - t}{k - t} U(k, \ell - 1)
\]
Theorem (M. and Moura)

If $n = k\ell$ is sufficiently large, a $t$-intersecting uniform partition system is no larger than a canonical system.

Use a counting method to find, bound the number of partitions in a non-canonical system. If $n$ is large, this bound is smaller than the size of a canonical set.

Proof.

1. Let $\mathcal{A}$ be a non-canonical intersecting set of partitions.

2. Assume $P = \{P_1, P_2, \ldots, P_\ell\} \in \mathcal{A}$.

3. Let $\mathcal{A}_i$ be all the partitions in $\mathcal{A}$ that contain $P_i$.
   since the system is intersecting every partitions will be in at least one $\mathcal{A}_i$

4. Bound the size of each $\mathcal{A}_i$,
   Bound uses that fact that it must intersect the partitions in $\mathcal{A}_j$.  

Since $\mathcal{A}$ is intersecting,
\[ |\mathcal{A}_i| \leq \left( \frac{\ell - 2}{t} \right) U(k, \ell - (t+1)) \]

If $(k \text{ is large relative to } \ell \text{ and } t)$ or $(k \geq t + 2 \text{ and } \ell \text{ large relative to } k \text{ and } t)$, then
\[ |\mathcal{A}| \leq \ell \left( \frac{\ell - 2}{t} \right) U(k, \ell - (t + 1)) < \frac{1}{\ell - t} \left( \frac{(\ell - t)k}{k} \right) U(k, \ell - (t + 1)) = U(k, \ell - t) \]

Proof is only difficult if $k = 2$ (perfect matching)
Perfect Matchings

If $k = 2$, the uniform $(2, \ell)$-partitions are **perfect matchings** of $2\ell$.

The number of perfect matchings is

$$(2\ell - 1)!! = (2\ell - 1)(2\ell - 3)(2\ell - 5) \cdots 3 \cdot 1$$

and the number of perfect matchings containing a fixed edge is

$$(2\ell - 3)!! = (2\ell - 3)(2\ell - 5) \cdots 3 \cdot 1.$$

The number of perfect matchings containing a $t$-fixed edges is $(2\ell - 2t - 1)!!$.

**Theorem**

*The size of the largest intersecting set of perfect matchings is $(2\ell - 3)!!$.***

We will see a totally different proof method for this.
Non-canonical Intersecting Perfect Matchings

**Conjecture**

For $\ell \geq 3t/2 + 1$ the size of the largest $t$-intersecting set of perfect matchings is 

$$(2(\ell - t) - 1)!!.$$

Recall the 3-sets, each contains at least two elements from $\{1, 2, 3\}$.

**Example**

If $t + 3 \leq \ell < 3t/2 + 1$ then there is a larger set of intersecting perfect matchings.

- Fix a set of $t + 2$ disjoint edges $\{e_1, e_2, \ldots, e_{t+2}\}$.
- Take all the perfect matchings that have at least $t + 1$ of these $t + 2$ edges.
- This set has size

$$\binom{t + 2}{t + 2}(2\ell - 2(t + 2) - 1)!! + \binom{t + 2}{t + 1}(2\ell - 2(t + 2))(2\ell - 2(t + 2) - 1)!!$$

which is larger than the canonical, if $\ell < 3t/2 + 1$

This construction works for the uniform $(k, \ell)$-partitions too, so the lower bound is needed.
Trivial Cases for Partial Intersection

1. Any two \((k, \ell)\)-uniform partitions are partially 1-intersecting.
2. If \(t = k\), then partially \(t\)-intersecting and intersecting are the same.

**Proposition**

If \(k > \ell(t - 1)\), then any two \((k, \ell)\)-partitions are partially \(t\)-intersecting.

If the parts are really big, any two partitions will be partially \(t\)-intersecting.

**Example**

Consider \(k = 5\), \(t = 3\), \(\ell = 2\) and \(n = 10\):

\[
\begin{align*}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 1 & 2 & 4 & 5 & 3 & 6 & 7 & 8 & 9 \\
0 & 1 & 5 & 6 & 9 & 2 & 3 & 4 & 7 & 8 & 9
\end{align*}
\]
Conjecture

If $k \leq \ell(t - 1)$, then the largest set of $t$-partially intersecting $(k, \ell)$-partitions is a canonical set of partially $t$-intersecting partitions.

Focus on partially 2-intersecting uniform partitions.

1. If $k = 2$, this is the intersecting perfect matchings.
2. Counting doesn’t work, we use a totally different method.
Derangement Graph

**Definition**

For any $k, \ell$ define the **partition derangement graph**, $\Gamma(k, \ell)$.

- The vertices are the uniform $(k, \ell)$-partitions.
- Vertices $P, Q \in \mathcal{U}(k, \ell)$ are adjacent if and only if $P$ and $Q$ are **not** partially 2-intersecting.

Can use any definition of intersection.

The graph $\Gamma_{(2,3)}$ (Image by Mahsa Shirazi).
Properties of Derangement Graphs

A **coclique** (independent set or stable set) in $\Gamma_{(k,\ell)}$, exactly if the set is a **partially $t$-intersecting set** of partitions.

**Graph Properties:**

1. The graph $\Gamma_{(k,\ell)}$ is **regular**; denote the degree by $d_{k,\ell}$
2. The derangement graph is **vertex-transitive**, the group $\text{Sym}(k\ell)$ acts transitively on the vertices of $\Gamma_{(k,\ell)}$.
   
   \text{Sym}(k\ell) \text{ permutes the elements in the partitions}

What are the maximum coclique in $\Gamma_{k,\ell}$?
Resolvable Designs

Definition

Suppose $B$ is a $t$-$(n, k, \lambda)$ design.

- Collection of $k$-sets from $\{1, \ldots, n\}$, every $t$-subset is in exactly $\lambda$ sets.
- A *parallel class* in $B$ is a collection of disjoint sets whose union is the $n$-set.
- A partition of $B$ into $r = \lambda \frac{(n-1)}{(t-1)} \frac{(k-1)}{(k-t)}$ parallel classes is called a *resolution*.
- A $t$-$(n, k, \lambda)$ design is *resolvable* if a resolution exists.

Example

Resolvable 2-$(9, 3, 1)$ Design:

- $123 \mid 456 \mid 789$ (orange)
- $147 \mid 258 \mid 369$ (red)
- $159 \mid 267 \mid 348$ (green)
- $168 \mid 249 \mid 357$ (blue)
Lemma

A resolvable \( t-(k\ell, k, 1) \) design is a maximum clique in \( \Gamma_{k,\ell} \) with size

\[
\frac{1 \binom{n}{t}}{\ell \binom{k}{t}}
\]

Theorem (Clique-Coclique Bound)

If \( X \) is a vertex-transitive graph and \( \alpha(X) \) is the size of the maximum coclique and \( \omega(X) \) the size of the maximum clique, then

\[
\alpha(X) \omega(X) \leq |V(X)|.
\]
Theorem

If there is a resolvable \( t-(k\ell, k, 1) \) design, then the canonical partially \( t \)-intersecting partitions are the largest intersecting sets.

Proof.

By the clique/clique bound, a coclique is no larger than

\[
\frac{U(k, \ell)}{1 - \ell} \left( \frac{k\ell - t}{k - t} \right) \left( \frac{k\ell - k}{k} \right) \left( \frac{k}{k} \right) = U(k, \ell - 1)
\]

Lemma (Meagher)

For \( n = 3k \) and \( k \) odd, partially \( 2 \)-intersecting uniform \( k \)-partition system is no larger than a canonical system.
Motivating Problem

Example

What is the largest set uniform (3,3)-partitions so that no two are 2-partially intersecting?

- This is the size of the largest clique in $\Gamma_{3,3}$.
- The number of uniform (3,3)-partitions is
  \[
  \frac{1}{3!} \binom{9}{3} \binom{6}{3} \binom{3}{3} = 280.
  \]
- A canonical 2-partially intersecting is a coclique of size $7 \times \frac{1}{2} \binom{6}{3} = 70$.
  \[
  1 2 \ast | \ast \ast \ast | \ast \ast \ast
  \]
- There is a clique of size 4.
  \[
  1 2 3 | 4 5 6 | 7 8 9, \quad 1 4 7 | 2 5 8 | 3 6 9, \quad 1 5 9 | 2 6 7 | 3 4 8, \quad 1 6 8 | 2 4 9 | 3 5 7
  \]
- This is maximum since
  \[
  70 \times 4 \leq \alpha(\Gamma_{3,3}) \omega(\Gamma_{3,3}) \leq |V(\Gamma_{3,3})| = 280.
  \]

A set in which no two are 2-partially intersecting corresponds to an **Orthogonal array**.
The eigenvalues of a graph are the eigenvalues of the adjacency matrix.

The largest eigenvalue of a $d$-regular graph is $d$.

**Theorem (Delsarte-Hoffman Ratio bound)**

Let $A$ be the adjacency matrix for a $d$-regular graph $X$ on vertex set $V(X)$. If the least eigenvalue of $A$ is $\tau$, then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}.$$ 

If equality holds for some coclique $S$ with characteristic vector $\nu_S$, then

$$\nu_S - \frac{|S|}{|V(X)|} \mathbf{1}$$

is an eigenvector with eigenvalue $\tau$.

The eigenspaces of $\Gamma_{k, \ell}$ are invariant under the action of $\text{Sym}(k\ell)$ and thus are a union of irreducible modules in the decomposition of

$$\text{ind} \left( 1_{\text{Sym}(k) \wr \text{Sym}(\ell)} \right)^{\text{Sym}(k\ell)}.$$ 

We say that an eigenvalue $\theta$ belongs to a module if the module is a subspace of the $\theta$-eigenspace.

If $\lambda \vdash n$ is an integer partition of $n$, then

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$$

(and $\lambda_1 + \lambda_2 + \cdots + \lambda_j = n$).

Each irreducible module of the symmetric group corresponds to an integer partition
What this looks like for the Perfect Matchings

1. The stabilizer of a single perfect matching is

\[ \text{Sym}(2) \wr \text{Sym}(\ell). \]

2. Since

\[ \text{ind} \left( 1_{\text{Sym}(2) \wr \text{Sym}(\ell)} \right)^{\text{Sym}(2\ell)} = \sum 2\lambda \]

where \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_j] \) is an integer partition of \( \ell \) and \( 2\lambda = [2\lambda_1, 2\lambda_2, \ldots, 2\lambda_j] \).

3. The eigenvalues of \( \Gamma_{k,\ell} \) can be found from the irreducible representations of \( \text{Sym}(2\ell) \), with the formula:

\[ \eta_{\phi}(A_{\ell}) = \frac{d_{\ell}}{|H|} \sum_{x_{\ell}} \sum_{h \in H} \phi(x_{\ell} h), \]

\( H = \text{Sym}(2) \wr \text{Sym}(\ell) \), \( d_{\ell} \) is the degree, \( x_{\ell} \) is a permutation in \( \text{Sym}(2\ell) \) so that the cosets \( H \) and \( x_{\ell} \) are intersecting partitions.

This formula is hard to use.

4. Srinivasan in "The perfect matching association scheme", has a better recursive algorithm.
Let $S$ be a coclique in a graph, and $V - S$ the remaining vertices.

If $S$ is a coclique in a $d$-regular graph

1. Counting edges gives $a = \frac{d|S|}{|V| - |S|}$.
2. The eigenvalues of the quotient graph are $d$ and $-a = -\frac{d|S|}{|V| - |S|}$.
3. Eigenvalues of the quotient matrix **interlace eigenvalues** of the adjacency matrix.
4. If $\{S, V \setminus S\}$ form an equitable partition, the $d$ and $-a$ are also eigenvalues of the adjacency matrix.
Let \( S \) be the set of all the perfect matchings with the edge \( 12 \), and \( V - S \) all the perfect matchings without the edge \( 12 \).

Every vertex in \( S \) is adjacent to no other vertices in \( S \), and \( d_{2,\ell} \) vertices in \( V - S \).

Every vertex in \( V - S \) is adjacent to \( a = \frac{d}{2\ell - 2} \) vertices in \( S \), and \( d - a \) other vertices in \( V - S \).

The quotient matrix is

\[
\begin{pmatrix}
0 & d \\
a & d - a
\end{pmatrix}
\]

\( d \) belongs to the representation \([2\ell]\) and \(-a\) belongs to \([2\ell - 2, 2]\).

**Lemma**

In \( \Gamma_{2,\ell} \), \( d \) is an eigenvalue of with multiplicity 1, (dimension of \([2\ell]\)), and \(-\frac{d}{2\ell - 2}\) is an eigenvalue with multiplicity at least \( \binom{2\ell}{2} - \binom{2\ell}{1} \) (dimension of \([2\ell - 2, 2]\)).
The trace of $A(\Gamma_2,\ell)^2$ is equal to the sum of the eigenvalues squared:

$$vd = \sum \lambda_i^2 m_i = d^2(1) + \left(-\frac{d}{2\ell-2}\right)^2 \left(\binom{2\ell}{2} - \binom{\ell}{1}\right) + \sum_{i \geq 2} \lambda_i^2 m_i$$

So for any other eigenvalue $\lambda_i$,

$$vd - d^2(1) + \left(-\frac{d}{2\ell-2}\right)^2 \left(\binom{2\ell}{2} - \binom{\ell}{1}\right) \geq \frac{\lambda_i^2 m_i}{m_i}$$

By considering the degrees of the irreducible representations of $\text{Sym}(2\ell)$, we can show that $-\frac{d}{2\ell-2}$ is the least eigenvalue.

Apply the ratio bound:

$$\alpha(\Gamma_2,\ell) \leq \frac{(2k - 1)!!}{1 - \frac{d}{2k-2}} = (2k - 3)!!$$
Perfect Matchings - List of Results

Theorem (Godsil and Meagher)

The size of the largest intersecting set of perfect matchings is \((2\ell - 3)!!\).

Characterization by the perfect matching polytope.

Theorem (Fallat, Meagher, Shirazi)

For \(\ell \geq 4\), the size of the largest 2-intersecting set of perfect matchings is \((2\ell - 5)!!\).

Theorem (Chase, Dafni, Filmus and Lindzey)

The only sets of 2-intersecting set of perfect matchings with size \((2\ell - 5)!!\) are the canonical intersecting sets.

Theorem (Lindzey)

For \(\ell\) sufficiently large, the size of the largest \(t\)-intersecting set of perfect matchings is \((2(\ell - t) - 1)!!\).
Eigenvalues for 2-Partially Intersecting \((k, \ell)\) Partitions

Master Plan:

1. find 3 specific eigenvalues of the graph,
2. show all other eigenvalues are smaller, (in absolute value)
3. apply the ratio bound.

Details:

1. The eigenspaces of \(\Gamma_{k,\ell}\) are invariant under the action of \(\text{Sym}(k\ell)\) and thus a union of irreducible modules in the decomposition of

\[
\text{ind} \left( 1_{\text{Sym}(k) \wr \text{Sym}(\ell)} \right)_{\text{Sym}(k\ell)}.
\]

2. For each irreducible representation in \(\text{ind} \left( 1_{\text{Sym}(k) \wr \text{Sym}(\ell)} \right)_{\text{Sym}(k\ell)}\) has a corresponding eigenvalue.

3. The irreducible representations \([k\ell], [k\ell - 2, 2]\) and \([k\ell - 3, 3]\) are included in \(\text{ind} \left( 1_{\text{Sym}(k) \wr \text{Sym}(\ell)} \right)_{\text{Sym}(k\ell)}\)

4. The irreducible representations \([1^{k\ell}], [k\ell - 1, 1], [2, 1^{k\ell - 2}], [2, 2, 1^{k\ell - 4}], [k\ell - 2, 1, 1], [3, 1^{k\ell - 3}], [2, 2, 2, 1^{k\ell - 6}]\) are not.

Proof by orbit counting.
Eigenvalues for 2-Partially Intersecting \((k, \ell)\) Partitions

**Theorem**

For any \(k, \ell\),

1. **The eigenvalue belonging to \([k\ell]\) is the degree** \(d = d_{k,\ell}\).

2. **The eigenvalue belonging to \([k\ell - 2, 2]\) is**
   \[
   \tau = -\frac{(k - 1)d}{k(\ell - 1)}.
   \]

3. **The eigenvalue belonging to \([k\ell - 3, 3]\) is**
   \[
   \theta = \frac{2(k - 1)(k - 2)d}{k^2(\ell - 1)(\ell - 2)}.
   \]

4. **For any other representation, the eigenvalue is smaller in absolute value** \(\tau\).

By the ratio bound, the maximum size of coclique in \(X_{k,\ell}\) is

\[
\frac{|V(\Gamma_{k,\ell})|}{1 - \frac{d}{\tau}} = \frac{U_{k,\ell}}{1 - \frac{d}{k(\ell - 1)}} = \frac{U_{k,\ell}}{1 + \frac{k(\ell - 1)d}{k - 1}} = \frac{U_{k,\ell}(k - 1)}{k\ell - 1} = \left(\frac{k\ell - 2}{k - 2}\right)U_{k,\ell - 1}.
\]
The action of a subgroup of the automorphism group on the partitions forms orbits. These orbits can be used to build a quotient graph.

Partition the vertices in the graph into orbits.

The quotient matrix

\[
\begin{pmatrix}
A & B & C \\
A & \begin{pmatrix} a & b & c \\
B & \begin{pmatrix} d & e & f \\
C & \begin{pmatrix} g & h & i \\
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\]
Quotient Graphs

Young’s subgroup $\text{Sym}(k\ell)$:
- Has one orbit, so the quotient graph is $\langle d \rangle$.
- This means $d$ is the eigenvalue corresponding to $[k\ell]$.

Young’s subgroup $\text{Sym}([k\ell - 2, 2]) = \text{Sym}(k\ell - 2) \times \text{Sym}(2)$
- Has two orbits: the partitions with 1 and 2 together in one part and the partitions where they are in two parts.
- The quotient matrix is the $2 \times 2$ matrix
  $$
  \begin{pmatrix}
  0 & d \\
  -\tau & d + \tau
  \end{pmatrix}
  $$
- The eigenvalues are $d$ and $\tau = -\frac{d(k-1)}{k(\ell-1)}$.
- This means $-\tau$ is the eigenvalue corresponding to $[k\ell - 2, 2]$. 

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Young’s subgroup $\text{Sym}([k\ell - 3, 3]) = \text{Sym}(k\ell - 3) \times \text{Sym}(3)$

- This group has three orbits: partitions where 1,2,3 are in one part, two parts or three parts.
- The quotient matrix is the $3 \times 3$ matrix

$$M = \begin{pmatrix}
0 & 0 & d \\
0 & a & d - a \\
b & c & d - b - c
\end{pmatrix}$$

- The eigenvalues are $d$, $-\tau$, $\theta$.
- $\text{tr}(M) = d - b - c + a = d - \tau + \theta$
- Counting edges between the orbits gives equations for $a,b,c$, then

$$\theta = \frac{2(k - 1)(k - 2)d}{k^2(\ell - 1)(\ell - 2)}.$$
Theorem

Assume $k\ell \geq 13$ and $k \geq 3$. Then the only partitions in the decomposition of

$$\text{ind } (1_{\text{Sym}(k) \wr \text{Sym}(\ell)})^{\text{Sym}(k\ell)}$$

with dimension less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$ are

$$\chi[k\ell], \; \chi[k\ell-2,2], \; \chi[k\ell-3,3].$$

Proof.

Use induction and the "branching rule".
**Bound on the Multiplicity**

**Theorem**

The eigenvalues $d$, $\tau$, and $\theta$ are the three largest, in absolute value, in the derangement graph. The smallest eigenvalue for $\Gamma_{(k,\ell)}$ is $\tau$.

**Proof.**

By squaring the adjacency matrix and taking the trace, we have

$$vd = d^2 + m \tau \tau^2 + m \theta \theta^2 + \sum_{i=2}^{j} m_i \lambda_i^2.$$

Hence for every $2 \leq i \leq j$, we have

$$vd - d^2 - m \tau \tau^2 - m \theta \theta^2 \geq m_i \lambda_i^2.$$

This gives an upper bound on $|\lambda_i|$ in terms of $d$.  

Karen Meagher: joint work with Chris Godsil, Lucia Moura, Mahsa Shirazi and Brett Stevens (University of Regina)
**Theorem (Bender)**

Let $M_{k,\ell}$ be the number of all $\ell \times \ell$ matrices with entries either 0 or 1, and row and columns sums equal to $k$. For positive integers $k, \ell$

$$\lim_{\ell \to \infty} \frac{(k!)^{2\ell}}{(k\ell)!} |M_{k,\ell}| = e^{-\frac{(k-1)^2}{2}}.$$

**Lemma**

For positive integers $k, \ell$ with $k \leq \ell$, let $d$ be the degree of $\Gamma_{(k,\ell)}$. Then

$$d = \frac{k! \ell}{\ell!} |M_{k,\ell}|.$$

Further, for a fixed integer $k$ with $k \geq 2$,

$$\lim_{\ell \to \infty} \frac{U(k, \ell)}{d} = e^{-\frac{(k-1)^2}{2}}.$$
Bound on Degree

We had

\[
\left( v d - d^2 - m \tau \left( \frac{d(k-1)}{k(\ell-1)} \right)^2 - m \theta \left( \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)} \right)^2 \right) \frac{1}{m_i} \geq |\lambda_i| 
\]

1. \( \lim_{\ell \to \infty} \frac{U(k,\ell)}{d} = e \frac{(k-1)^2}{2} \) gives an upper bound for \( \lambda_i \).

2. If another eigenvalue is larger than \( \tau \), in absolute value, that eigenvalue has to have a multiplicity smaller than \( \binom{k\ell}{3} - \binom{k\ell}{2} \).

3. Only \( d, \tau \) or \( \theta \) have multiplicity this small.

4. So \( \tau \) is the least eigenvalue of \( \Gamma_{k,\ell} \).

5. Apply the ratio bound.
Theorem

Fix an integer \( k \geq 3 \). For \( \ell \) sufficiently large, the largest set of partially 2-intersecting uniform \((k, \ell)\)-partitions has size \( \binom{k \ell - 2}{k - 2} U_{k, \ell - 1} \).

Conjecture

For \( k \geq 3 \) and \( \ell \) sufficiently large, the only sets of partially 2-intersecting \((k, \ell)\)-partitions with size \( \binom{k \ell - 2}{k - 2} U_{k, \ell - 1} \) are the sets \( S_{i,j} \).

With a more tedious calculation on the degree approximation we get the follow:

Theorem

For \( k = 3 \) and all \( \ell \geq 3 \) the largest set of partially 2-intersecting uniform partitions has size

\[
(3\ell - 2) U_{3, \ell - 1}.
\]
Future Work

There are two obvious questions:

**Question**
Can a non-canonical 2-partially intersecting set also have the maximum size?

**Question**
Can this method be extended \( t \)-partially intersecting uniform partitions for large values of \( t \)?