

Topology of isotropy orbits

Inaugural-Dissertation zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität Augsburg

vorgelegt von
Augustin-Liviu Mare
aus Cluj-Napoca (Rumänien)

Augsburg 1997

Berichterstatter:
Prof. Dr. J.-H. Eschenburg
Prof.Dr. E. Heintze

Tag der mündlichen Prüfung: 07.07.1998

Vorwort

Die vorliegende Arbeit verbindet zwei mathematische Disziplinen: Geometrie und Topologie. Die Topologie beschreibt, wie sich ein Raum aus einfachen Bausteinen, "Zellen" genannt, zusammensetzen lässt. Ein Mittel hierfür ist eine in den 30'ger Jahren von M. Morse entwickelte Theorie, die sich am Beispiel einer (unebenen) Fläche F erklären lässt: Man betrachtet nicht ganz F auf einmal, sondern nur den Teil F^h von F , der unterhalb einer bestimmten Höhe h liegt, und man untersucht, wie sich F^h verändert, wenn wir h grösser werden lassen. Nur wenn es auf F Berge, Täler oder Pässe mit Höhe h gibt, sogenannte kritische Punkte, wird sich der topologische Typ von F^h dabei ändern und eine weitere Zelle aufnehmen. Mit der Morsetheorie erhält man also eine Zerlegung einer Fläche oder eines Raumes in Zellen. Man möchte aber noch mehr wissen, nämlich wie die Zellen sich untereinander schneiden ("Cup-Produkt"), und dafür ist die Morsetheorie i.a. nicht ausreichend. Augustin-Liviu Mare beschreibt in der vorliegenden Arbeit, wie diese Aufgabe für eine bestimmte Klasse von Räumen durch genauere Untersuchung ihrer Geometrie dennoch gelöst werden kann.

Die untersuchten Räume sind die Untermannigfaltigkeiten mit konstanten Hauptkrümmungen, sicher eine der interessantesten Klassen von Räumen in der ganzen Geometrie; zu ihnen gehören die Bahnen der Drehgruppen von Symmetrischen Räumen ("Isotropie-Orbits"). Ein solcher Raum M zeichnet sich u.a. dadurch aus, dass jeder seiner Punkte auf einer Reihe von Kugelflächen liegt, die ganz in M enthalten sind. Die Untersuchung dieser Kugelfamilien und der von ihnen gebildeten mehrfachen Sphärenbündel bildet die wichtigste Grundlage der Arbeit. Für Sphärenbündel lässt sich das Cup-Produkt nämlich mit dem Satz von Leray-Hirsch berechnen und dann auf M übertragen. Dies konnte in früheren Arbeiten nur in sehr speziellen Fällen durchgeführt werden; durch Verfeinerung der Methoden gelingt Herrn Mare die Lösung im allgemeinen Fall. Als ein Nebenprodukt enthält die Arbeit auch eine neue Sichtweise und eine Verallgemeinerung der klassischen Schubertzellen in Grassmann-Mannigfaltigkeiten.

Die Arbeit stellt keine hohen Anforderungen an das Vorwissen des Lesers. Wer Interesse an der Verbindung von geometrischen und topologischen Methoden hat, wird diese Arbeit mit grossem Gewinn lesen.

Augsburg, im Mai 1998

Professor Dr. Jost-Hinrich Eschenburg
Institut für Mathematik
Universität Augsburg

Zusammenfassung

Das erste Zugang zur Topologie von Liegruppen und homogenen Räumen war vielleicht gemacht von A. Borel am Anfang der fünfziger Jahre. Die Homologie oder Kohomologie solcher Räume waren bekommen von ihm mit Hilfe hauptsächlich der Theorie der Spectralsequenzen von Prinzipalbündeln (siehe z.B. [3]). Ein total anderes Zugang zur gleichen Thema ist enteckt geworden ein paar Jahre später von R. Bott und H. Samelson. Sie haben bemerkt, dass die Theorie von Morse ein sehr nützliches Instrument für die Nachforschung der Homologie und sogar Kohomologie einer breiten Klasse von Homogenen Räumen sein könnte, nämlich von Orbiten so- genannter variational vollständigen Operationen kompakter Liegruppen. Ähnliche Ideen waren später benützt von Hsiang, Palais und Terng für die Nachforschung der Homologie isoparametrischer Untermannigfaltigkeiten in Euklidischen Räumen. Sie haben auch versucht, die Methoden von Bott und Samelson zu erweitern, um die Kohomologie solcher Räume zu bestimmen, aber sie haben das nur teilweise geschafft. Das Hauptziel unserer Arbeit ist, diese Aufgabe vollständig aufzuführen. Wir sind fähig, Formeln die die Kohomologie geben, nicht nur für isoparametrische Untermannigfaltigkeiten, sondern auch für beliebige Elemente einer isoparametrischen Blätterung (d.h. für fokale Mannigfaltigkeiten).

Summary

Maybe the first direct approach to the topology of Lie groups and homogeneous spaces was made by A. Borel at the beginning of the 50's. Homology or cohomology of such spaces were obtained by him by using as basic tool the theory of spectral sequences of principal bundles (see for instance [3]). A totally different approach of the same subject was given a few years later by R. Bott and H. Samelson. They noticed that the theory of Morse could be a very useful instrument in investigating the homology and even the cohomology of a large class of homogeneous spaces, namely the orbits of the so-called variational complete actions of compact Lie groups. Similar ideas were later used by Hsiang, Palais and Terng in investigating the homology of isoparametric submanifolds in euclidean spaces (see [19]). They also tried to extend the methods of Bott and Samelson in order to determine the cohomology of such spaces, but they succeeded only partially. The main goal of our present paper is to achieve completely this task. We are able to give formulas which describe the cohomology not only for isoparametric submanifolds, but also of arbitrary elements of an isoparametric foliation (i.e. for focal manifolds).

Mathematics Subject Classification: 57T15, 57R70, 55U40

Augsburg, im Mai 1998

Augustin-Liviu Mare

Contents

1	Homology of isoparametric submanifolds	1
1.1	Elements of Morse theory	1
1.2	Basic facts about isoparametric submanifolds	7
1.3	The homology module of M	15
1.4	Words, galleries and their homotopy	18
1.5	Homology of M via homotopy of galleries	21
1.6	The homology of M_v	34
1.7	Schubert cycles on Grassmannians as cycles of Bott-Samelson type	40
1.8	Schubert cycles on Grassmannians as projected HPT-cycles	47
2	Cohomology of isoparametric submanifolds	55
2.1	The representation induced by W on $H_{lower}(M)$	56
2.2	Homology and cohomology of F_p	73
2.3	The ring $H^*(M_v, \mathcal{R})$	78
2.4	Isoparametric hypersurfaces in spheres	90
2.5	Uniform multiplicity 1 or 2	97
2.6	The Weyl group of B_k type and different multiplicities	101
2.7	The Weyl group of F_4 type and different multiplicities	103
3	Lusternik-Schnirelmann category of isoparametric submani- folds	105
3.1	Basic results and primar estimations	105
3.2	Adjoint orbits of dimension less than 6 and their F.-category .	110
3.3	Open questions	113

Appendix.....	108
Bibliography.....	111
Index.....	113

Introduction

The main idea of the theory of Morse is to relate the topology of a finite or infinite dimensional manifold M and the critical data of a nondegenerate real function defined on it (namely the set of its critical points and their indexes). Roughly speaking, we are going along the real axis and follow the changes which occur in the topology of the sublevel sets during this process. By the two so-called deformation lemmas, one notice no change as long no critical points are met in this way and we simply have to add a cell of dimension k after passing through a critical point of index k . At the "end" (recall that M is not necessarily compact) of this process we obtain the space M as a cellular complex . A lot of things about the homology module of M can be proved in this situation : for instance, the inequalities of Morse say that the rank of the homology module must be at least the number of critical points of f . But it is in general impossible to see in this way which is exactly the rank of $H_*(M)$.

It follows easily from some elementary reasonings that the latter one equals the number of critical points (i.e. f is *perfect*), provided that at each critical point, the cell added to the lower sublevel can be extended to a homological cycle which is closed in the upper sublevel: the origins of this idea go back to the early papers of M. Morse (see for instance [34], Satz, p.26). One of the major contributions of the classical work [5] of Bott and Samelson is the remark that the extension we were talking about can be obtained as a cycle "hanging over" the corresponding critical point. We will call it *cycle of Bott-Samelson type* (see section 1.1 of our paper for the exact definition). The two authors above considered then loop spaces over orbits of so-called *variationally complete* actions together with the energy function defined on this space. At each critical point (i.e. geodesic segment) they were able to construct a cycle hanging over it and to determine in this way completely the homology of this loop space. It turned out then that the isotropy action of any symmetric space is variationally complete and everything can be said in this way about the $\mathbb{Z}_2$ -homology of the orbits of this representation.

The idea to prove that certain nondegenerate functions are perfect by using cycles of Bott-Samelson type seems very tempting by its naturality.

But it is in general hard to obtain them and, at least when nondegenerate distance functions are involved, they probably exist only in some very particular situations. In his paper [39] G.Thorbergsson proved that any complete Dupin hypersurface embedded in a simply connected space form is *taut* (i.e. any non-degenerate distance function is perfect) by an ingenious construction of cycles of Bott-Samelson type.

After defining in 1985 the notion of isoparametric submanifold in a Euclidean space (see [38]), Terng, this time together with Hsiang and Palais will use in [19] the same construction as Thorbergsson in order to prove that these submanifolds are also taut embedded. The cycle that the first three authors mentioned above associated to any critical point of a distance function (and which we will call in our paper *HPT-cycle*) has a very simple structure: it is an iterated sphere bundle which admits a section at each iteration. Having in mind that any isotropy orbit of a symmetric space is element of an isoparametric foliation, it can be estimated that all results of [5] concerning the homology of isotropy orbits were recovered by Hsiang, Palais and Terng without using any more the Morse theory for loop spaces, but directly for the submanifolds which are involved and the corresponding distance functions. The basic tool of their paper is the using of certain integrable distributions whose leaves are full spheres which, at their turn, can be employed in constructing the cycles of Bott-Samelson type.

After obtaining the homology module for isotropy orbits, Bott and Samelson considered the particular case of adjoint orbits of compact Lie groups and determined completely their integral cohomology. Hsiang, Palais and Terng tried in the last section of their up-mentioned paper to extend this method, but they succeeded only partially, in some particular cases. In fact, by the homogeneity theorem of Thorbergsson (which says that, with some isolated exceptions, any isoparametric foliation is in fact the orbital foliation of the isotropy representation for a certain symmetric space) it can be seen that the cohomological estimations of [19] bring nothing new: the only spaces whose cohomology is exactly evaluated are just the adjoint orbits of simple Lie groups. The real value of section 6 of [19] stays, from our viewpoint, not in the direct evaluations made there, but in its qualitative side: concretely, the authors obtained the cohomology of isoparametric submanifolds of uniform and even multiplicity by reducing the problem to well-known representatives of this class, namely the adjoint orbits of Lie groups (as we mentioned above, it turned out later that the class of isoparametric submanifolds with uniform even multiplicity reduces almost entirely to the adjoint orbits, but the idea

of "prototype" remains).

The main goal of our paper is to go further in the direction initiated in [19], in the new light of Thorbergsson's homogeneity theorem, and to give in this way a natural continuation of Bott and Samelson's paper. Namely, we will obtain precise formulas for the $\mathcal{R}$ -cohomology rings of all elements of irreducible isoparametric foliations in Euclidean spaces, where $\mathcal{R}$ can be $\mathbb{Z}$ or $\mathbb{Z}_2$ if the multiplicities of M are greater than 1, and only $\mathbb{Z}_2$ when there exists a multiplicity equal to 1. Our basic tool will be still the notion of cycle of Bott-Samelson type, which is indispensable to us. In particular, the only reason because we are not able to calculate the integral cohomology when a multiplicity equals 1, is that in this case the HPT-cycles are not necessarily orientable, so they cannot constitute cycles of Bott-Samelson type over the ring $\mathbb{Z}$. In other words, nothing more can be obtained about the cohomology of the spaces we are referred here by employing only the classical theory of Morse. We must mention here the approach of Kocherlakota (see [22]) who obtained the integral homology for all isotropy orbits of symmetric spaces, by using the so-called Witten-Morse complex. But the problem of finding a general result which describes the integral *cohomology* of these orbits remains, for the moment, open.

We will also try to consider the isoparametric submanifolds from the point of view of other invariants: the Lusternik-Schnirelmann category and the so-called F.-category (in the sense of Takens). The exact value of these two invariants will be obtained only for adjoint orbits of compact connected semisimple Lie groups. Upper or lower bounds in some other situations (most notably, for isoparametric hypersurfaces in spheres) will be obtained.

There are two major goals that we intend to reach in the first chapter:

a) to show how and why could be the HPT-cycles replaced by what we called F-cycles;

b) to obtain a closer relationship between the homology of an isoparametric submanifold and the homology of its focal manifolds, in terms of so-called *projected cycles* (subsidiarily, we will obtain the well-known Schubert cycles on Grassmannians as projected HPT-cycles).

The essence of the paper is concentrated in section 3 of the second chapter. An essential ingredient will be the representation induced in homology by the action of the Weyl group on the isoparametric submanifold, described in section 2.1. We consider then an arbitrary F-cycle F_p associated to the critical point of maximal index: it is the total space of an iterated sphere

bundle with sections at each level, and it will be relatively simple to obtain its homology and cohomology. The evaluations obtained in section 2.3 will be then exemplified in the particular case of the isoparametric foliations of the sphere. The last three sections of Ch.II are intended to use the "prototype principle" we mentioned above in order to obtain a complete picture of the $\mathcal{R}$ -cohomology rings for *all* irreducible isoparametric submanifolds. We will succeed that up to some exceptions.

The main result of Ch.III gives the F.-category of an arbitrary adjoint orbit of a compact connected semisimple Lie group (see Cor.3.3.6), and states that it equals its Lusternik-Schnirelmann category. So in this very particular situation, the answer to the general conjecture which states the equality of these two invariants for any closed finite-dimensional manifold is positive. Finally, we consider the isoparametric submanifolds of type B_2 (that represents a very important class, because only in this situation non-homogeneous examples are known, and no classification has been obtained so far): we prove that the L.S.-category must be 4 or 5 and we conjecture that, at least when both multiplicities are different and greater than 1, the value of this invariant is 4.

I would like, also on this way, to express my deep gratitude to Prof. Dr. Jost Eschenburg for the enormous amount of energy and time he spent in helping me to elaborate this theses. By stimulating conversations, scientific and moral support, he offered me generously his help during all the time when I prepared the paper. Almost all main ideas was suggested by him. I must mention that section 1.7 reproduces almost entirely ideas that he exposed during one of his lectures and he communicated me later.

I owe thanks to the Graduiertenkolleg "Nichtlineare Probleme in Analysis, Geometrie und Physik" from the University of Augsburg, which admitted me as member and supported financially the research during the time when the present paper was elaborated. For helpful discussions, bibliographical informations and moral support, I must thank Prof. Dr. Ernst Heintze. I am grateful to Prof. Dr. Kazumoto Kojima, who indicated me many of the bibliographical sources which I needed in section 2.5. One of the main ideas of the paper, namely the use of galleries in the construction of what I called later F-cycles, was suggested to me by Robert Bock; I am obliged to him for that. I also thank Andreas Kollross for the help offered during the search of the exact expression for the principal isotropy orbit presented in section 2.7.

Chapter 1

Homology of isoparametric submanifolds

1.1 Elements of Morse theory

The theory of Morse in its classical form will constitute one of our basic tools in investigating the topology of isoparametric submanifolds. In the following we will try to summarize some notions and results used during this chapter. The main sources are [32] and [26]. Because of the context of our paper it will be enough to restrict our attention exclusively to closed finite-dimensional manifolds, even if a theory in a much more general context is available, as can be seen for instance in [32].

Let's fix such a manifold M^n and take a real function f on M . A point $x \in M$ is called *critical* if the differential $(df)_x$ vanishes identically. Otherwise, x will be called *regular*. A real number is a *critical* value if it is image of a critical point; if not, we call it a *regular* value. The corresponding levels are also called *critical*, respectively *regular*.

If p is now a critical point of f , then there exists a uniquely determined symmetric bilinear form $Hess(f)_p$ on $T_p M$ (called the *Hessian* of f at p), such that for any two vector fields X and Y we have $Hess(f)_p(X_p, Y_p) = X_p(Yf)$. The *index* of the critical point p is simply the index of this bilinear form. A critical point is said *nondegenerate* if the associated Hessian is nondegenerate.

One knows that the expression of f around a regular point is well-determined. The question is what happens at a critical point. If this is a nondegenerate one, we still have a complete answer, given by the lemma of Morse:

Theorem 1.1.1 (Morse lemma) *Around any critical point p , nondegenerate of index k , there exists a local parametrisation of M of the form (V, ψ) , with $\psi(0) = p$ and*

$$(f \circ \psi)(x_1, x_2, \dots, x_n) = f(p) - (x_1)^2 - \dots - (x_k)^2 + (x_{k+1})^2 + \dots + (x_n)^2.$$

Consequently, the nondegenerate critical points are isolated in the set of all critical points. We will restrict ourselves exclusively to functions with only nondegenerate critical points, so-called *Morse functions*. Their critical sets will be, of course, finite. By technical reasons and without loss of generality we will suppose during this section that the critical levels corresponding to any two distinct critical points are also distinct.

For any number a , denote $M_a = \{p \in M : f(p) \leq a\}$. The basic idea of Morse theory is to describe the topology of M "inductively" going along the real axis and looking at the evolutions occurred in the topology of the sublevel sets M_a , $a \in \mathbb{R}$. Two steps are needed to see that:

Theorem 1.1.2 *Suppose $a \leq b$ are finite and there are no critical points in $f^{-1}[a, b]$. Then:*

- (i) M_a is diffeomorphic to M_b .
- (ii) $M_a \subseteq M_b$ is a strong deformation retract.
- (iii) The inclusion map $M_a \hookrightarrow M_b$ is a homotopy equivalence.

In other words, nothing changes in the topology of M when passing only through regular levels. But there are also a certain number of critical levels. Let's now see what happens after passing a such one. If p is a critical point of index k with $c = f(p)$, fix (V, ψ) a Morse chart and consider the k -cell

$$e^k = \{p \in \psi^{-1}(V) : x_1(p)^2 + \dots + x_k(p)^2 \leq \epsilon, x_{k+1}(p) = \dots = x_n(p) = 0\},$$

where $\epsilon > 0$ is enough small, such that the ball $B(0, \epsilon)$ is contained in V and no critical point different from p lies in $f^{-1}([c - \epsilon, c + \epsilon])$.

Theorem 1.1.3 *The set M_c is a strong deformation retract of $M_{c+\epsilon}$ and $M_{c-\epsilon} \cup e^k$ is a deformation retract of M_c .*

In other words, from the point of view of the homotopy type, after passing through the critical level corresponding to an index k critical point, a k -cell is attached. A classical consequence says that any closed manifold has the homotopy type of a finite CW-complex: by Whitney theorem we can embed

the manifold in a Euclidean space, and then take into account that almost all associated distance functions are Morse functions.

Consider now $a < b$ two finite numbers such that in $f^{-1}(a, b)$ there exists just one critical point p . We have just seen that, up to a homotopy equivalence, M_b is the same as $M_a \cup e^k$, where e^k is a k -cell, in other words one has to add a cell in the CW-complex. The question is now what consequences could that have for the evolution of the homology module. Consider to this end the usual singular homology with coefficients in an arbitrary ring $\mathcal{R}$ and the long exact sequence in homology for the pair (M_a, M_b) , related to the long exact homology sequence of the pair $(e^k, \partial e^k)$ through the inclusion map $G : (e^k, \partial e^k) \rightarrow (M_b, M_a)$. In the following diagram, the map $G_* : H_*(e^k, \partial e^k) \rightarrow H_*(M_a, M_b)$ is always an isomorphism.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H_m(e^k, \partial e^k) & \longrightarrow & H_{m-1}(\partial e^k) & \longrightarrow & H_{m-1}(e^k) \longrightarrow H_{m-1}(e^k, \partial e^k) \longrightarrow \cdots \\
& & \downarrow G_* & & \downarrow G_* & & \downarrow G_* \\
\cdots & \longrightarrow & H_m(M_b, M_a) & \xrightarrow{\partial_m} & H_{m-1}(M_a) & \xrightarrow{i_{m-1}} & H_{m-1}(M_b) \xrightarrow{j_{m-1}} H_{m-1}(M_b, M_a) \rightarrow \cdots
\end{array}$$

One also knows that $H_m(e^k, \partial e^k)$ equals $\mathcal{R}$ for $m = 0$ and $m = k$, and $\{0\}$ otherwise. Denote by $[e^k, \partial e^k]$ the fundamental cycle of the pair $(e^k, \partial e^k)$. Since i_m is an isomorphism for $m \neq k$ and $m \neq k - 1$, we can consider the following exact sequence:

$$0 \rightarrow H_k(M_a) \xrightarrow{i_k} H_k(M_b) \xrightarrow{j_k} \mathcal{R}[e^k, \partial e^k] \xrightarrow{\partial} H_{k-1}(M_a) \xrightarrow{i_{k-1}} H_{k-1}(M_b) \rightarrow 0$$

Distinguish the following two situations:

1. The boundary ∂e_k is not homologous to 0 in M_a , in other words the map ∂ from the sequence above is non identically zero (see the picture below).

When $\mathcal{R}$ is a field, this implies ∂ injective, so $j_k = 0$ and the following exact sequence is obtained:

$$0 \longrightarrow \mathcal{R}[e^k, \partial e^k] \longrightarrow H_{k-1}(M_a) \longrightarrow H_{k-1}(M_b) \longrightarrow 0.$$

Because this sequence splits, it follows

$$H_{k-1}(M_a) \simeq H_{k-1}(M_b) \oplus \mathcal{R}[e^k, \partial e^k].$$

In the same time, i_k is an isomorphism, so the only change in the homology module arises at the level $k - 1$, namely the rank of this level decreases by 1. Intuitively, we can feel that something "wrong" happened. A similar

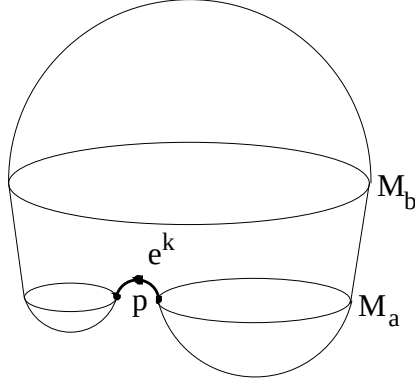


Figure 1.1: Bad Morse theory

analysis can be, of course, made for an arbitrary ring $\mathcal{R}$. We will not get into these details; our only intention was to convince that that the present case case is an "unreasonable" one.

2. The $(k-1)$ -cycle ∂e_k is homologous to 0 in M_a (geometrically, the phenomenon can be easily visualized in the following classical picture) .

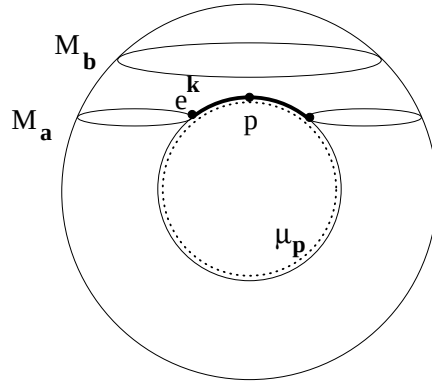


Figure 1.2: Good Morse theory

This simply means $\partial = 0$ and from the splitting exact sequence

$$0 \longrightarrow H_k(M_a) \xrightarrow{i_k} H_k(M_b) \longrightarrow \mathcal{R}[e_k, \partial e_k] \longrightarrow 0$$

one deduces i_k injective and $H_k(M_b) = H_k(M_a) \oplus \mathcal{R}[e_k, \partial e_k]$. One notes in

this situation a "positive" change, that in an increase of $\text{rank} H_*$ of one unit, at the level k . It seems very clear that, as long a Morse function presents at any of its critical points a such behaviour, a *good* Morse theory can be developed in investigating the homology of M .

Remark and definition. The second situation is equivalent to the existence of a cycle $[\mu_p] \in H_*(M_b)$ whose image is $[e_k, \partial e_k]$. Geometrically, μ_p is a full cycle in M_b (i.e. $\partial \mu_p = 0$) which extends the relative cycle $[e^k, \partial e^k] \in H_*(M_b, M_a)$. If it exists, a such cycle will be called a *linking cycle* over $\mathcal{R}$. The critical point p is said of *linking type* over $\mathcal{R}$ if there exists a linking cycle at p . We say that f is of *linking type* over $\mathcal{R}$ if all its critical points are of linking type.

It turns out easily from the previous considerations how a Morse function of linking type on M determines the homology of M :

Theorem 1.1.4 *Let $f : M \rightarrow \mathbb{R}$ be a Morse function of linking type over $\mathcal{R}$. For each critical point p of f , denote by $k(p)$ its index and by $[\mu_p] \in H_{k(p)}(M_{f(p)+\epsilon}) \subset H_{k(p)}(M)$ the corresponding linking cycle. Then for any $a \in \mathbb{R}$, the module $H_*(M_a, \mathcal{R})$ is freely generated by the set $\{[\mu_p] : p \in C(f), f(p) \leq a\}$. Particularly, the module $H_*(M, \mathcal{R})$ is freely generated by the set of all linking cycles.*

Morse functions of linking type have a nice behaviour from the point of view of the Morse inequalities. Recall that if $f : M^n \rightarrow \mathbb{R}$ is an arbitrary Morse function, $c_j = c_j(f)$ the number of index j critical points of f and $b_j = b_j(M) = \text{rank} H_j(M, \mathcal{R})$, $\mathcal{R}$ being an arbitrary ring, then:

Theorem 1.1.5 (Morse inequalities) *For any $0 \leq k \leq n$ holds:*

$$c_k - c_{k-1} + \dots + (-1)^k c_0 \geq b_k - b_{k-1} + \dots + (-1)^k b_0.$$

Moreover, for $k = n$ the corresponding inequality is actually an equality:

$$\chi(M) := b_0 - b_1 + \dots + (-1)^n b_n = c_0 - c_1 + \dots + (-1)^n c_n.$$

Consequently, for any $1 \leq j \leq n$, $c_j(f) \geq b_j(M)$.

A Morse function is called $\mathcal{R}$ -*perfect* if it realizes equality in all Morse inequalities, in other words, it has a minimal number of critical points. From Th.1.1.4 above follows :

Corollary 1.1.6 *Any Morse function of linking type over $\mathcal{R}$ is $\mathcal{R}$ -perfect.*

Now, that we just have seen how advantageous a Morse function of linking type could be, a natural question is how often a such function can occur. Everything reduces to the obtention of the linking cycles: we will describe in the sequel a way to get them. Concretely, the linking cycles will be obtained as so-called Bott-Samelson cycles, in the form defined for the first time in [19]. From now on, $\mathcal{R}$ will be always $\mathbb{Z}$ or $\mathbb{Z}_2$.

Definition If $p \in M$ is an index k critical point, a cycle of *Bott-Samelson type* over $\mathcal{R}$ at p is a pair (Y, φ) , where Y is a closed, k -dimensional, $\mathcal{R}$ -orientable manifold and $\varphi : Y \rightarrow M$ such that $f \circ \varphi$ has a unique maximum, which is nondegenerate, at $y_0 = \varphi^{-1}(p)$.

Note that in this case φ is an immersion at y_0 and $(d\varphi)_{y_0}$ maps TY_{y_0} isomorphically onto the negative space of f at p .

Theorem 1.1.7 *If (Y, φ) is a cycle of Bott-Samelson type over $\mathcal{R}$ at p , then $\varphi_*[Y]$ is a linking cycle over $\mathcal{R}$ at p .*

PROOF. (see [32]).

Corollary 1.1.8 *A Morse function $f : M \rightarrow \mathbb{R}$ any of whose critical points admits a cycle of Bott-Samelson type is a linking type function.*

So as long as we are able to construct cycles of Bott-Samelson type, a good Morse theory could be developed. In concrete applications, the only problem will be only to obtain the cycles explicitly: an ingenious construction, belonging to Hsiang, Palais and Terng (see [19]) furnishes cycles of this type for nondegenerate distance functions associated to isoparametric submanifolds in Euclidean spaces (we will come back with more details in section 3).

A last aspect we would like to consider in this preliminary section concerns the Morse theory of distance functions for submanifolds of Euclidean spaces. Let $X : M^n \rightarrow \mathbb{R}^{n+k}$ be an immersion of the closed manifold M . One can immediately associate the so-called *endpoint map* $Y : \nu(M) \rightarrow \mathbb{R}^{n+k}$, $Y(x, v) = x + v$, for any $x \in M$ and $v \in \nu(M)_x$. For any $v \in C^\infty(\nu(M))$ and any $x_0 \in M$ one associates the *shape operator* $A_v : TM_{x_0} \rightarrow TM_{x_0}$, $A_v(u) = -((D_u v)(x_0))^T$, $u \in TM_{x_0}$, where D denotes the usual derivative in $\mathbb{R}^{n+k}$. This is a linear selfadjoint map and its (real) eigenvalues are the

principal curvatures at x_0 in direction v . On the normal bundle $\nu(M)$ one considers the *normal connection* ∇^ν , which associates to $u \in C^\infty(TM)$ and $v \in C^\infty(\nu(M))$ the normal field $(\nabla^\nu)_u v = (D_u v)^\nu$, i.e. the orthogonal projection of $D_u v$ on $\nu(M)$. Directly from the definition follows:

$$D_u v = -A_v(u) + (\nabla^\nu)_u v.$$

We call $a \in \mathbb{R}^{n+k}$ *focal point* of multiplicity $m \geq 1$ with respect to $p \in M$ if $\dim(\text{Ker } dY_{(p,v)}) = m$, provided $(p, v) \in \nu(M)$ with $Y(p, v) = a$.

Theorem 1.1.9 *Let $p \in M$, $v \in \nu(M)_p$ and $a = Y(p, v) = p + v \in \mathbb{R}^{n+k}$. Then:*

- (i) *p is a critical point of the distance function $f_a : M \rightarrow \mathbb{R}$, $f_a(x) = \frac{1}{2}\|x - a\|^2$, $x \in M$.*
- (ii) *the Hessian of f at p is $(\text{Hess}f)_p = I - A_v$, by assimilating the selfadjoint linear operators on TM_p with symmetric bilinear functionals.*
- (iii) *p is nondegenerate if and only if a is nonfocal with respect to p .*
- (iv) *the index of f_a at p equals the number of focal points with respect to p situated on the line segment $[pa]$, each counted with its multiplicity.*

From (iii) is now clear that f_a is a Morse function iff a is non-focal, i.e. a is a regular value of Y ; by Sard theorem, that happens for almost all $a \in \mathbb{R}^{n+k}$. In studying immersions in Euclidean spaces, the notion of tautness is fundamental; for the moment we will restrict ourselves to the definition, and the anticipative remark that isoparametric submanifolds offer the most illustrative examples of taut-immersions.

Definition The immersion $X : M \rightarrow \mathbb{R}^{n+k}$ is called *taut* if any distance function which is nondegenerate is $\mathbb{Z}_2$ -perfect.

1.2 Basic facts about isoparametric submanifolds

We will give in the sequel a brief exposition of the main notions and results concerning isoparametric submanifolds in Euclidean spaces. Detailed treatments of this theory can be founded in [38], [19] or [32]. Even if it's maybe too soon, we must say from the very beginning that the objects involved here are mainly homogeneous, namely isotropy orbits of symmetric spaces. Such orbits are the objects of study in [5], and it's one of the main goals of our

paper to show that the investigations initiated there can be continued by using the techniques developed by Hsiang, Palais and Terng in [19].

Definition An immersed submanifold $X : M^n \rightarrow \mathbb{R}^{n+k}$ is called *isoparametric* if the normal bundle is flat (with respect to the normal connection) and the principal curvatures in the direction of any parallel normal field are constant, as values and multiplicities.

The definition assumes flatness only locally, but it turns out that M is globally flat. Even from the local flatness it follows that for any $x \in M$, the set $\{A_v : v \in \nu(M)_x\}$ is a commutative family of selfadjoint endomorphisms of TM_x and so it determines a decomposition of TM_x as direct sum of the common eigenspaces $E_1(x), E_2(x), \dots, E_p(x)$. The normal vectors $v_1, v_2, \dots, v_p$ which satisfy the following relations:

(a) $A_v|_{E_i} = \langle v, v_i \rangle id_{E_i}$, for any $v \in \nu(M)_x$.

(b) if $i \neq j$, there exists $v \in \nu(M)_x$ with $\langle v, v_i \rangle \neq \langle v, v_j \rangle$. arise in a natural way. The situation can be immediately globalized, in the sense that one can consider the smooth distributions $E_1, E_2, \dots, E_p$ on M (called the *curvature distributions*) and the parallel normal vector fields $v_1, v_2, \dots, v_p$ (called the *curvature normals*) such that (a) holds at any point of M . The numbers $m_i = \text{rank } E_i$, $1 \leq i \leq p$, are called the *multiplicities* of M .

Without loss of generality we will make the following assumptions, always available during this paper :

1. M is complete.
2. M is full, in the sense that it is contained in no proper affine subspace of $\mathbb{R}^{n+k}$. Equivalently, the curvature normals $v_1, v_2, \dots, v_p$ at any point x generate the normal space $\nu(M)_x$. The number k is now uniquely determined and we will call it the *rank* of M .
3. M is compact, or equivalently, all curvature normals are nonzero. Moreover, M is a closed submanifold of the sphere S^{n+k-1} .

Any distribution E_i , $1 \leq i \leq p$, is integrable. Its leaf through the point x is a standard sphere of radius $\frac{1}{\|v_i\|}$ and center $x + \frac{v_i(x)}{\|v_i\|^2}$, contained in the affine plane $x + E_i(x) \oplus \nu(M)_x$ and called the *i-th curvature sphere* through x .

One of the basic ingredients of this theory is the very particular structure of the focal set (i.e. the set of all focal points) of the embedding $X : M \rightarrow \mathbb{R}^{n+k}$. If Γ is this focal set, and $\nu_x(M) = x + \nu(M)_x$ denotes the affine normal space at x , then $\Gamma \cap \nu_x(M)$ is the union of hyperplanes $l_1(x), l_2(x), \dots, l_p(x)$, where $l_i(x)$ is perpendicular to $v_i(x)$ at the distance $\frac{1}{\|v_i\|}$ from x .

It turns out that any distribution E_i , $1 \leq i \leq p$ is integrable and moreover,

its leaf through an arbitrary point $x \in M$, denoted by $S_i(x)$, is a round sphere, called the *curvature sphere* through x . It is a full sphere in $x + E_i(x) + \nu(M)_x$, which focalizes at $l_i(x)$.

The reflection R_i^x in $\nu_x(M)$ with respect to $l_i(x)$ leaves the set of all focal hyperplanes invariant, hence the group W^x generated by $\{R_i^x : 1 \leq i \leq p\}$ is a finite reflection group of rank k . Because the normal holonomy is trivial, the same construction can be parallelly transported from x to any other point of M . In this way, we associated to M a finite reflection group W , which doesn't depend any more on the choosing of x . We will see later that W is in fact a crystallographic group; that's the reason because we call it the *Weylgroup* of M .

Also consider for any $i \in \{1, 2, \dots, p\}$ the map $\varphi_i : M \rightarrow M$ which associates to any x the antipodal $\varphi_i(x)$ of x in the sphere $S_i(x)$. Of course, any φ_i is an involutive automorphism of M . We can express $\varphi_i(x)$ analytically as

$$\varphi_i(x) = x + \frac{2v_i(x)}{\|v_i\|^2} = R_i^x(x). \quad (1.1)$$

The group generated by $\{\varphi_i\}_{1 \leq i \leq p}$ is naturally isomorphic to W , and one obtains in this way an action of W on M . An interesting geometric particularity of this action is given by :

Theorem 1.2.1 *For any $x \in M$ we have $M \cap \nu_x(M) = W.x$. If $x_1 \in W.x$, then the affine normal spaces at x and x_1 are the same.*

We consider now in $\nu_x(M)$ the chamber (i.e. closure of a connected component of $\nu_x(M) \setminus \cup\{l_i(x) : 1 \leq i \leq p\}$) which contains x , denoted by $\Delta(x)$. The group W^x is generated by the set of reflections at the walls of $\Delta(x)$. Moreover, the walls of $\Delta(x)$ are always "the same", in the sense that the set $\{1 \leq i \leq p : l_i(x) \text{ bounds } \Delta(x)\}$ is independent of $x \in M$. Choose the indices in such a way, that $l_1(x), l_2(x), \dots, l_k(x)$ bound $\Delta(x)$, for any $x \in M$. The indices $1, 2, \dots, k$ and also the corresponding curvature spheres will be called *fundamental*.

Let now v be a parallel normal vector field, such that $x + v(x) \in \Delta(x)$. Associate first the *end-point map* $\pi_v : M \rightarrow \mathbb{R}^{n+k}$, $y \mapsto y + v(y)$, $y \in M$ and then:

$$\begin{aligned} I(v) &= \{1 \leq i \leq k : x + v(x) \in l_i(x)\}, \\ J(v) &= \{1 \leq j \leq p : x + v(x) \in l_j(x)\} = \{1 \leq j \leq p : \pi_v(S_j(x)) \text{ reduces to a point}\}, \end{aligned}$$

$k_v = \text{card} I(v)$, $n_v = \sum_{j \in J(v)} m_j$,
 $\xi_v^x = (\bigcap_{i \in I(v)} l_i(x))^\perp_{\nu M_x}$, i.e. the orthogonal complement of $\bigcap_{i \in I(v)} l_i(x)$ in νM_x ;

finally, W_v^x denotes the subgroup of W^x generated by $\{R_i^x : i \in I(v)\}$, i.e. the stabilizer of $x + v(x)$ in W (notice that $I(v)$, $J(v)$ and implicitly k_v and n_v do not depend on the choosing of x). We do not exclude the case when $x + v(x)$ lies in the interior of $\Delta(x)$, situation that can be equivalently expressed by any of the following conditions: $I(v) = \phi$, or $J(v) = \phi$, or $k_v = 0$, or $n_v = 0$, or $W_v^x = W$.

Theorem 1.2.2 (i) *The map $\pi_v : M \rightarrow \mathbb{R}^{n+k}$ has constant rank $n - n_v$ and its image M_v is an embedded submanifold in $\mathbb{R}^{n+k}$. The tangent, respectively normal spaces at an arbitrary point $x^* = \pi_v(x) \in M_v$ are:*

$$\nu(M_v)_{x^*} = \nu(M)_x + \sum_{j \in J(v)} E_j(x),$$

respectively

$$(TM_v)_{x^*} = \sum_{j \notin J(v)} E_j(x).$$

(ii) *Regarded as a map from M to M_v , π_v is a fiber bundle. If $S_{x,v}$ is the connected component through x of the fiber $\pi_v^{-1}(\pi_v(x))$, then $S_{x,v}$ is an n_v -dimensional isoparametric submanifold of rank k_v in $\xi_v^x + \sum_{j \in J(v)} E_j(x)$ having ξ_v^x as normal plane at x . Its focal hyperplanes at x are $l_j(x) \cap \xi_v^x$, $j \in J(v)$ and the Weyl group is W_v^x . The curvature distributions of $S_{x,v}$ and consequently the curvature spheres and multiplicities are naturally inherited from M .*

The manifold $S_{x,v}$ will be called the *slice* corresponding to v through x . As a particular case of this theorem, consider the situation when $I(v) = \phi$, in other words when no focalisation occurs: then π_v maps M diffeomorphically onto the isoparametric submanifold M_v . Such M_v will be called a *parallel manifold*. In the opposite case, when v focalizes M at least once, we call M_v a *focal manifold*. The family $\{M_v : x + v(x) \in \Delta(x)\}$ is called the *isoparametric foliation* of $\mathbb{R}^{n+k}$ induced by M .

Another particular case is given by choosing i, j two different fundamental indices and taking a normal vector v at x such that $x + v(x)$ belongs to $l_i(x)$, $l_j(x)$ and to no other fundamental focal hyperplane, that is

$I(v) = \{i, j\}$. For the rank 2 isoparametric submanifold $S_{x,v}$, $l_i(x) \cap \xi_v^x$ and $l_j(x) \cap \xi_v^x$ are fundamental focal lines regarded from x . But the angle between $l_i(x)$ and $l_j(x)$ is the same as the one between the two lines above, and the latter one, by a result of Münzner (see [29]), must be $\frac{\pi}{p}$, with $p \in \{2, 3, 4, 6\}$. It follows the result that we have already anticipated, namely that W is a crystallographic group.

The second part of the section concerns an important class of isoparametric submanifolds, namely the principal isotropy orbits of symmetric spaces. Consider $X = G/K$ a simply connected symmetric space of noncompact type. Essentially we have here a simply connected semisimple noncompact Lie group G together with an involutive automorphism θ whose fixed point set K is a connected, maximal compact subgroup of G , or equivalently, in terms of Lie algebras, with the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G , respectively K and $\mathfrak{p} = TX_{eK}$. By [18], Prop.7.4, Ch.III, Cartan decomposition means that the following conditions are fulfilled:

- a) $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$;
- b) the Killing form B of $\mathfrak{g}$ is strictly negative definite on $\mathfrak{k}$ and strictly positive definite on $\mathfrak{p}$.

It follows that K acting on $\mathfrak{g}$ through the adjoint representation leaves $\mathfrak{p}$ invariant. The representation of K on $\mathfrak{p}$ induced in this way is the so-called *isotropy representation* of X , and its orbits are called simply *isotropy orbits*.

Consider now a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ and $x \in \mathfrak{a}$ a regular point (i.e. the centralizer of x in $\mathfrak{p}$ reduces to $\mathfrak{a}$). The corresponding isotropy orbit $M = Ad(K).x$ is called *principal* and will be regarded as a compact, embedded submanifold of the Euclidean space $(\mathfrak{p}, B|_{\mathfrak{p}})$. The tangent, respectively normal spaces of M at an arbitrary point $Ad(k).x$ are:

$$T(M)_{Ad(k).x} = [Ad(k).x, \mathfrak{k}], \quad \nu(M)_{Ad(k).x} = Ad(k).\mathfrak{a}, \quad (1.2)$$

so the codimension of M in $\mathfrak{p}$ equals the rank of X . We associate to any $v \in \mathfrak{a}$ the normal vector field $\hat{v}$ defined by $\hat{v}(Ad(k).x) = Ad(k).v$, for any $k \in K$. The derivative of this field in the direction $[\xi, x] \in TM_x$, $\xi \in \mathfrak{k}$, is given by:

$$D[\xi, x]\hat{v} = (d\hat{v})_x([\xi, x]) = [\xi, v]. \quad (1.3)$$

It follows that $\hat{v}$ is parallel with respect to ∇^ν and consequently $(\nu(M), \nabla^\nu)$ is flat. The shape operator at x is described by $A_v : TM_x \rightarrow TM_x$, $[\xi, x] \mapsto -[\xi, v]$, $\xi \in \mathfrak{k}$.

Recall now that a linear real functional $\lambda : \mathfrak{a} \rightarrow \mathbb{R}$ is a root if the space $\mathfrak{g}_\lambda = \{z \in \mathfrak{g} : [x, z] = \lambda(x)z, \forall x \in \mathfrak{a}\}$ is not zero. If $\Lambda \subset \mathfrak{a}^*$ denotes the set of nonzero roots and $\mathfrak{k}_0$ the commutator of $\mathfrak{a}$ in $\mathfrak{k}$, then $\mathfrak{g}$ decomposes as:

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{k}_0 + \sum_{\lambda \in \Lambda} \mathfrak{g}_\lambda. \quad (1.4)$$

Since $B|_{\mathfrak{a}}$ is strictly positive definite, we can associate to any $\lambda \in \Lambda$ a vector $H_\lambda \in \mathfrak{a}$ uniquely determined by $B(x, H_\lambda) = \lambda(x)$, for any $x \in \mathfrak{a}$. Identifying $\mathfrak{a}^*$ with $\mathfrak{a}$, we can associate to any $\lambda \in \Lambda$ (the same as H_λ) the reflection s_λ of $\mathfrak{a}$ in the hyperplane $l_\lambda = \{x \in \mathfrak{a} : \lambda(x) = 0\}$ and obtain in this way on Λ a structure of root system in $\mathfrak{a}$, which is not necessarily reduced. The Weyl group $W = W(X)$ corresponding to $\mathfrak{a}$ is defined as

$$W = N_K(\mathfrak{a})/C_K(\mathfrak{a}) = \{k \in K : Ad(k).\mathfrak{a} \subseteq \mathfrak{a}\} / \{k \in K : Ad(k).x = x, \forall x \in \mathfrak{a}\}.$$

Via the adjoint representation Ad , W acts faithfully on $\mathfrak{a}$; it is generated by the reflections s_λ , $\lambda \in \Lambda$.

If $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$ denotes the complexification of $\mathfrak{g}$, then $\mathfrak{g}^c = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_\mathbb{C}$. Denote by Δ the set of nonzero roots of $\mathfrak{g}^c$ with respect to a maximal abelian subalgebra $\mathfrak{a}' \supseteq i\mathfrak{a}$. Any such root is an $\mathbb{R}$ -linear functional on $\mathfrak{a}'$ with values in $i\mathbb{R}$ and all elements of Λ are, up to a multiplication by i , restrictions to $\mathfrak{a}$ of elements of Δ . The *multiplicity* n_λ of an arbitrary element $\lambda \in \Lambda$ represents the number of elements of Δ which restrict to λ on $\mathfrak{a}$.

Consider now $S = \{\gamma_1, \dots, \gamma_k\}$ the simple root system consisting of the roots which correspond to the walls of the Weyl chamber containing x and associate Λ^+ the set of positive roots. Also distinguish in Λ the subsets Λ_0 and Λ_0^+ of indivisible, respectively positive indivisible roots. Together with $\mathfrak{g}_\lambda$, one associates to any $\lambda \in \Lambda$ the space $\mathfrak{k}_\lambda = \{z \in \mathfrak{k} : (ad x)^2(z) = \lambda(x)^2 z, \forall x \in \mathfrak{a}\} = (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}) \cap \mathfrak{k}$. The direct decomposition $\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \Lambda^+} \mathfrak{k}_\lambda$ gives for any regular $x \in \mathfrak{a}$:

$$[x, \mathfrak{k}] = \sum_{\lambda \in \Lambda^+} [x, \mathfrak{k}_\lambda]. \quad (1.5)$$

Notice that if $\lambda \in \Lambda$, then the space $\mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$ is a subalgebra of $\mathfrak{k}$.

Let's now come back to our principal isotropy orbit $M = \text{Ad}(K).x$, $x \in \mathfrak{a}$ regular. Take $v \in \mathfrak{a}$ a normal vector, $\hat{v}$ the parallel normal field induced by v and notice that for any $y_\lambda \in \mathfrak{k}_\lambda$ holds $A_{\hat{v}}[y_\lambda, x] = -[y_\lambda, v] = -\frac{\lambda(v)}{\lambda(x)}[y_\lambda, x]$, because $y_\lambda \in \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}$. The decomposition (1.5) rewritten as

$$[x, \mathfrak{k}] = \sum_{\lambda \in \Lambda_0^+} [x, \mathfrak{k}_\lambda] \oplus [x, \mathfrak{k}_{2\lambda}] \quad (1.6)$$

is just the eigenspace decomposition of TM_x corresponding to any shape operator $A_{\hat{v}}$ and its eigenvalues $-\frac{\lambda(v)}{\lambda(x)}$, $\lambda \in \Lambda_0^+$.

It is now clear that M is an isoparametric submanifold of $(\mathfrak{p}, B|_{\mathfrak{p}})$. A synthesis of the previous considerations concerning the structure of M as an isoparametric submanifold is made in the following theorem:

Theorem 1.2.3 *Let $M = \text{Ad}(K).x$ be a principal isotropy orbit of the rank k symmetric space $X = G/K$. Then:*

- (i) *M is an isoparametric submanifold of rank k in $\mathfrak{p}$ whose corresponding isoparametric foliation is just the orbital foliation of the isotropy representation of K on $\mathfrak{p}$.*
- (ii) *the curvature distributions at x are $E_\lambda(x) = [x, \mathfrak{k}_\lambda] \oplus [x, \mathfrak{k}_{2\lambda}]$, with the induced multiplicity $m_\lambda = n_\lambda + n_{2\lambda}$, $\lambda \in \Lambda_0^+$.*
- (iii) *the curvature normals at x are $-\frac{1}{\lambda(x)}H_\lambda$, $\lambda \in \Lambda_0^+$.*
- (iv) *the curvature spheres through x are the orbits $\text{Ad}(K_\lambda).x$, where K_λ denotes the closed subgroup of K corresponding to the Lie subalgebra $\mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$.*
- (v) *the Weyl group of M is $W(X)$ and its action on M through antipodal maps is the same as the action of W on $\text{Ad}(K).x$ from the right. More precisely, for any $\lambda \in \Lambda_0^+$ and any $k \in K$ holds*

$$\varphi_\lambda(\text{Ad}(k).x) = \text{Ad}(k).(s_\lambda(x)).$$

- (vi) *M is irreducible if and only if the symmetric space X is irreducible. In this situation, we have at most two distinct values of the multiplicities.*

The last statement follows immediatly from the list of all possible marked Dynkin diagrams of irreducible symmetric spaces. What we call here the *marked Dynkin diagram* is simply obtained from the Dynkin diagram of the root system Λ by if we figure out for each node $\gamma_i \in S$ the value m_{γ_i} of its corresponding multiplicity. Directly from [18], Table VI, p. 532 and some elementary calculations can be deduced:

Lemma 1.2.4 *The only possible marked Dynkin diagrams for irreducible symmetric spaces of rank greater than 2 are these given in Table 1 from Appendix.*

Among the isoparametric families in Euclidean spaces, the orbital foliations of isotropy representations play an essential role, as can be seen in the following theorem:

Theorem 1.2.5 *Any compact, connected, complete, irreducible and full isoparametric submanifold of rank at least 3 is a principal isotropy orbit of an irreducible symmetric space.*

Three different proofs of this theorem are known so far. These were done, successively, by Thorbergsson (see [40]), Olmos (see [31]) and Heintze and Liu (see [16]). In fact, the last two authors proved in [16] a more general result, namely the homogeneity for infinite dimensional isoparametric submanifolds in Hilbert spaces. In the theorem above it is essential that M has rank greater than 2, in the sense that there exist examples of rank 2 isoparametric submanifolds in Euclidean spaces (equivalently, isoparametric hypersurfaces in spheres) which are not homogeneous (see [13]). Anyway, by [29], the number of distinct multiplicities for isoparametric hypersurfaces in spheres is still at most 2. Consequently:

Corollary 1.2.6 *The number of distinct multiplicities of an irreducible isoparametric submanifold is at most 2.*

We will usually denote their values by m_1 , m_2 or, when it happens to be equal, by m .

Remarks 1. Instead of the symmetric space $X = G/K$ of noncompact type, the dual symmetric space $X' = G'/K$ could have been considered. The only difference consists in the fact that the Lie algebra $\mathfrak{g}'$ of G' decomposes this time as $\mathfrak{g}' = \mathfrak{k} \oplus i\mathfrak{p}$, hence up to a multiplication by i , the treatment is equivalent.

2. It is a classical fact that if G is a compact connected semisimple Lie group, then $G \times G/\Delta(G)$ is a symmetric space of compact type with respect to the automorphism $(g_1, g_2) \mapsto (g_2, g_1)$ of $G \times G$. The two terms of the corresponding decomposition of the Lie algebra $\mathfrak{g} \times \mathfrak{g}$ of $G \times G$ are this time $\mathfrak{k} = \{(x, x) : x \in \mathfrak{g}\}$ respectively $\mathfrak{p} = \{(x, -x) : x \in \mathfrak{g}\}$. If $\mathfrak{t} \subset \mathfrak{g}$ is a maximal abelian subalgebra, then $\mathfrak{a} = \{(x, -x) : x \in \mathfrak{t}\}$ is a maximal abelian subspace of $\mathfrak{p}$. If Δ denotes the set of roots of G , then the roots of the symmetric space

$G \times G/G$ are of the form $\tilde{\alpha} : \mathfrak{a} \rightarrow i\mathbb{R}$, $(x, -x) \mapsto \alpha(x)$, $x \in \mathfrak{t}$, with $\alpha \in \Delta$. The only roots of $G \times G$ which restrict to $\tilde{\alpha}$ are $\alpha + 0$ and $0 + \alpha$. The isotropy representation is this time

$$\Delta(G) \times \mathfrak{p} \rightarrow \mathfrak{p}, \quad (g, g).(x, -x) = (Ad(g).x, Ad(g).(-x)),$$

for $g \in G$ and $x \in \mathfrak{t}$, that is nothing but the adjoint representation of G . So we can finally conclude that any principal adjoint orbit of G is an isoparametric submanifold of uniform multiplicity 2 in $\mathfrak{g}$.

1.3 The homology module of M

Let $X : M^n \rightarrow \mathbb{R}^{n+k}$ be a compact, connected, complete and full isoparametric submanifold. In the sequel, we would like to analyse this embedding from the point of view of the notions presented in our first section: the tautness will be proved in terms of cycles of Bott-Samelson type for nondegenerate distance functions. The only aim of the present section is to reproduce the construction of these cycles, preceded by some preliminary considerations, from [19].

We begin with the description of the critical set for nondegenerate distance functions.:

Theorem 1.3.1 *Let $a \in \mathbb{R}^{n+k}$ be an arbitrary point, $f_a : M \rightarrow \mathbb{R}$ the corresponding distance function and $C(f_a)$ its critical set. Let also be $x_0 \in C(f_a)$, i.e. $a \in \nu_{x_0}(M)$.*

- a) The W -orbit of x_0 is contained in $C(f_a)$.*
- b) If a is W^{x_0} -regular (or, equivalently, a nonfocal point of the embedding), then f_a is a Morse function and $C(f_a) = W.x_0$.*
- c) Under the same assumptions as at the point b), the index of f_a at any $x \in W.x_0$ equals the number of focal hyperplanes from $\nu_{x_0}(M)$ intersected by the line segment $[ax]$, each counted with its multiplicity.*

PROOF (see [19], Th.5.1)

□

The construction of the cycles of Bott-Samelson type for f_a at x_0 belonging to Hsiang, Palais and Terng is described in the following theorem (the ideas of this construction will be later important for us; that's why we consider useful to reproduce also the proof of the theorem, cf [32]:

Theorem 1.3.2 *Let $a \in \mathbb{R}^{n+k}$ and $x_0 \in M$ such that $a \in \nu_{x_0}(M)$ is W^{x_0} -regular. Consider the focal hyperplanes intersected by the line segment $[ax_0]$ be $l_{i_1}(x_0), l_{i_2}(x_0), \dots, l_{i_r}(x_0)$, going from a to x_0 . Then:*

- a) *The negative space of f_a at x_0 is $\oplus\{E_{i_j}(x_0) : 1 \leq j \leq r\}$.*
- b) *Take $N = \{(y_1, \dots, y_r) \in M \times \dots \times M : y_1 \in S_{i_1}(x_0), y_2 \in S_{i_2}(y_1), \dots, y_r \in S_{i_r}(y_{r-1})\}$ and $u : N \rightarrow M, (y_1, \dots, y_r) \mapsto y_r$. Then (N, u) is a cycle of Bott-Samelson type over $\mathcal{R}$ for f_a at the critical point x_0 . The ring $\mathcal{R}$ is $\mathbb{Z}$ if all values of the multiplicities are greater than 2, respectively $\mathbb{Z}_2$ otherwise.*

PROOF a) Remember that $(Hess f_a)_{x_0} = I - A_{a-x_0}$. If v denotes the normal vector $a - x_0$, then $A_v|_{E_i} = \langle a - x_0, v_i \rangle id_{E_i}$, so that the negative space of the Hessian at x_0 is the direct sum of those E_i for which $\langle a - x_0, v_i \rangle > 1$. Geometrically, this latter fact means that $[ax_0]$ crosses the hyperplane $l_i(x_0)$.

b) It will be sufficient to prove the following:

- (i) $y_0 = (x_0, \dots, x_0)$ is the unique maximum point of $f_a \circ u$.
- (ii) $(du)_{y_0}$ maps TN_{y_0} isomorphically onto the negative space of f_a at x_0 .
- (iii) if m_{i_j} are all greater than 1, then N is orientable.

The point (ii) is quite clear, since $TN_{y_0} = \times_{1 \leq j \leq r} E_{i_j}(x_0)$ and $(du)_{y_0}$ is the natural identification of the latter product with the direct sum of the same spaces. Also (iii) follows immediately from the fact that N is the total space of an iterated bundle whose fibers and initial base space are spheres of dimensions m_{i_j} and consequently is simply connected. In order to understand the idea of proving (i) it will be enough to take $r=3$, $i_1 = 1$, $i_2=2$ and $i_3=3$.

For an arbitrary $y = (y_1, y_2, y_3) \in N \setminus \{y_0\}$ we shall prove that $(f \circ u)(y) < (f \circ u)(y_0)$, i.e. $d(a, y_3) < d(a, x_0)$. In order to do that, consider $z_j = [ax_0] \cap l_j(x_0)$, $1 \leq j \leq 3$. Take $a_1 = y_1 + v(y_1)$ and look at the intersections of $[a_1 y_1]$ with $l_1(y_1)$, $l_2(y_1)$ and $l_3(y_1)$: $z_j^{(1)} = [a_1 y_1] \cap l_j(y_1)$, $1 \leq j \leq 3$. Since the parallel transport from x_0 to y_1 in the normal bundle of the sphere $S_1(x_0)$ is an orthogonal transformation which fixes $l_1(x_0)$ pointwise, it follows $l_1(y_1) = l_1(x_0)$, so $z_1^{(1)} = z_1$ and also $|a_1 z_1| = |a z_1|$, $|z_1^{(1)} z_2^{(1)}| = |z_1 z_2|$, $|z_2^{(1)} z_3^{(1)}| = |z_2 z_3|$, $|z_3^{(1)} y_1| = |z_3 x_0|$. The same reasoning is then used for the parallel transport from y_1 to y_2 in the normal bundle of $S_2(y_1)$. We consider this time $a_2 = y_2 + v(y_2)$, $[a_2 y_2] \cap l_j(y_2) = z_j^{(2)}$, $1 \leq j \leq 3$ and notice that $l_2(y_2) = l_2(y_1)$, so $z_2^{(2)} = z_2(1)$, and also $|a_2 z_1^{(2)}| = |a_1 z_1^{(1)}|$, $|z_1^{(2)} z_2^{(2)}| = |z_1^{(1)} z_2^{(1)}|$, $|z_2^{(2)} z_3^{(2)}| = |z_2^{(1)} z_3^{(1)}|$, $|z_3^{(2)} y_2| = |z_3^{(1)} y_1|$. Finally we invoke the parallel transport from y_2 to y_3 in $\nu(S_3(y_2))$ in order to see that if $a_3 = y_3 + v(y_3)$, $[a_3 y_3] \cap l_j(y_3) = z_j^{(3)}$, $1 \leq j \leq 3$, then $l_3(y_3) = l_3(y_2)$, hence $z_3^{(3)} = z_3^{(2)}$, and also $|a_3 z_1^{(3)}| = |a_2 z_1^{(2)}|$,

$|z_1^{(3)} z_2^{(3)}| = |z_1^{(2)} z_2^{(2)}|$, $|z_2^{(3)} z_3^{(3)}| = |z_2^{(2)} z_3^{(2)}|$, $|z_3^{(3)} y_3| = |z_3^{(2)} y_2|$. The key of this construction is to notice that the union $[az_1] \cup [z_1^{(1)} z_2^{(1)}] \cup [z_2^{(2)} z_3^{(2)}] \cup [z_3^{(3)} y_3]$ is a polygonal line in $\mathbb{R}^{n+k}$ which joins a to y_3 , having the same length as $[ax_0]$. It follows $d(a, y_3) \leq d(a, x_0)$. We could have equality only if the broken line above is the same as $[ax_0]$, in other words, at each step of the previous construction nothing "moves", i.e. $y_1 = x_0$, $y_2 = y_1$, $y_3 = y_2$, so $y = (x_0, x_0, x_0)$. □

Definition The cycle (N, u) constructed above is called the *HPT-cycle* of f_a at x_0 .

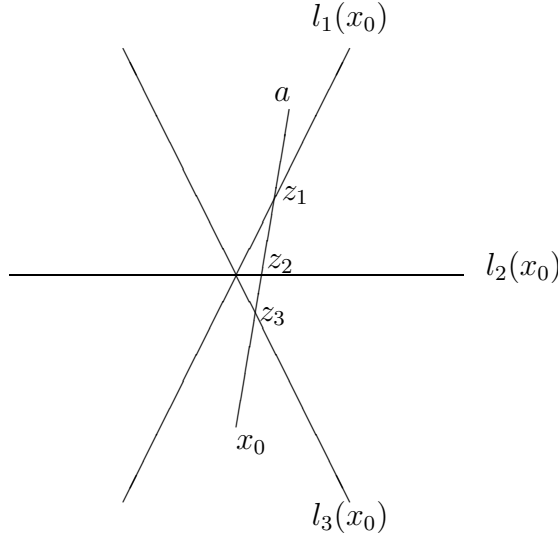


Figure 1.3

By Corollary 1.1.8, everything is known about the $\mathcal{R}$ -homology of M :

Corollary 1.3.3 a) If all multiplicities of M are greater than 1, then:

- (i) M is simply connected;
- (ii) $H_*(M, \mathbb{Z})$ has no torsion;
- (iii) the dimension of $H_i(M, \mathbb{Z})$ equals the number of index i critical points of f_a ;
- (iv) $\dim H_*(M, \mathbb{Z}) = |W|$.

b) The manifold M is always taut-embedded in $\mathbb{R}^{n+k}$ and $\dim H_*(M, \mathbb{Z}_2) = |W|$.

The central point of the section was to present the so-called HPT-cycles. Although a very ingenious construction, originating in the classical work [5] of Bott and Samelson, we estimate that some arbitrariness exists here. One could ask for instance what happens when the segment $[ax_0]$ contains the origin 0, i.e. the common intersection of all focal hyperplanes from $\nu_{x_0}(M)$? The construction makes no sense any more. We could try to go further and improve the situation by moving a without leaving its Weyl chamber: even if the hyperplanes crossed by $[ax_0]$ are in this way well-determined, they are not always intersected in the same order, and there exists for the moment no reason to believe that this order doesn't influence the corresponding cycle. In order to obtain a certain control of this phenomenon, we will need a description of HPT-cycles in terms of galleries and words. A minimal background regarding these latter notions will be presented in our next section.

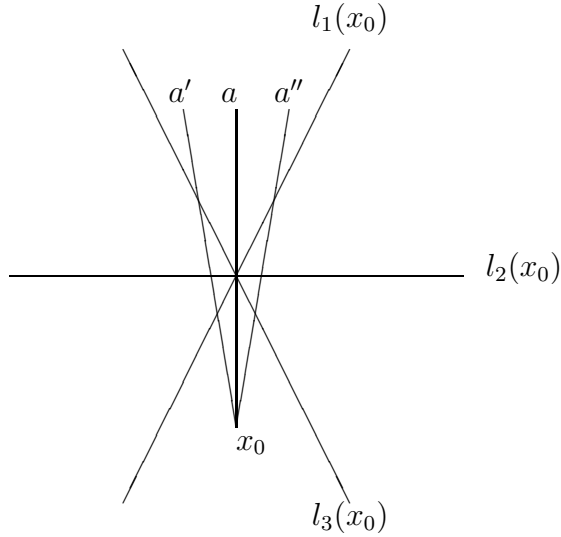


Figure 1.4

1.4 Words, galleries and their homotopy

Following [33] we will present in the sequel some elementary facts of the theory of buildings: the notion of building will be not explicitly involved and the general context will not exceed the classical theory of finite (spherical)

Weyl groups (see for instance [20]).

Consider an irreducible finite Weyl group W on the Euclidean space $\mathbb{R}^k$ and denote by $\mathcal{H}$ the family of reflection hyperplanes. Among the Weyl chambers, distinguish arbitrarily a certain one, denote it by c_0 and call it the *fundamental chamber*. Since W acts simply transitively on the set of all chambers, we will frequently identify the latter ones to the elements of W as follows: c_0 is the same as the unit e and an arbitrary chamber c the same as the element of W which maps c_0 to c . Denote by $S = \{s_1, \dots, s_k\}$ the subset of W consisting of the reflections at the walls of c_0 . Then (W, S) is a finite *crystallographic Coxeter system*, i.e. S generates W and the only relations between generators are of the form $\text{ord}(s_i s_j) = m_{ij}$, $1 \leq i, j \leq k$, with $m_{ii} = 1$ and $m_{ij} \in \{2, 3, 4, 6\}$ for $i \neq j$. Denote by $\mathcal{I}_k$ the set $\{1, 2, \dots, k\}$ and call its elements *fundamental indices*. Two chambers c_1 and c_2 are called *i-adjacent*, $i \in \mathcal{I}_k$, if $c_1 = c_2 s_i$. In order to understand the geometric meaning of the adjacency relation we "colour" the walls of all chambers, by using the colours $1, 2, \dots, k$ as follows: a wall p of a certain chamber c is coloured by j if and only if c , regarded as an element of W , maps the wall s_j of the chamber c_0 onto p . The colouring is evidently well-defined; two chambers c_1 and c_2 are then *j-adjacent* if and only if one of them can be obtained from the other by a reflection at a wall of colour j (in the figure below, for instance, c_1 and c_2 are 2-adjacent).

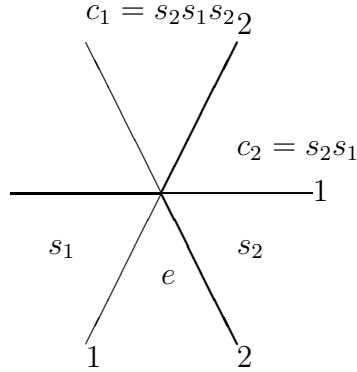


Figure 1.5

The *words* are defined as elements of the free monoid generated by $\mathcal{I}_k$. Associate to any word $f = i_1 i_2 \dots i_r$ the element $s_f = s_{i_1} s_{i_2} \dots s_{i_r}$ of W . A

gallery of type f starting at c_0 is a system of chambers $(c_0, c_1, \dots, c_r)$ such that c_j is i_j -adjacent with c_{j-1} , for any $1 \leq j \leq r$. Any such gallery is uniquely determined by its type. We will usually identify the set $\mathcal{G}$ of galleries starting at c_0 with the set of words. Notice that there exists a gallery of type f from $\mathcal{G}$ which ends at w if and only if $w = s_f$. The *length* of a gallery of type $i_1 i_2 \dots i_r$ is by definition r . Associate to any $w \in W$ the *distance* $d(e, w)$ from e to w as the minimal length of a gallery from $\mathcal{G}$ ending at w , and also its *length* $l(w)$ as the minimal number of factors that a decomposition of w like a product of simple reflections can have. Clearly $l(w) = d(e, w)$. A gallery from $\mathcal{G}$ which ends at a certain w is said *minimal* if its length is minimal, namely $d(e, w)$.

The following remark will play an important role in the understanding of our future applications. It can be founded in K.S. Brown's book [7] and we would like to reproduce it *ad literam*: "A minimal gallery should be thought as the combinatorial analogue of a geodesic: any minimal gallery from c_0 to c crosses precisely the same elements of $\mathcal{H}$ as a straight line segment joining a point of c_0 to a point of c (of course, the minimal gallery need not cross these walls in the same order as the line segment)".

Let's now associate to any pair $i, j \in \mathcal{I}_k$ the word $p(i, j) = \underbrace{ijij \dots}_{m_{ij} \text{ factors}}$.

If, for instance, $m_{ij} = 3$, then $p(i, j) = ijij$ and $p(j, i) = jiji$. In general we have $s_{p(i, j)} = s_{p(j, i)}$. An *elementary homotopy of words* is a transformation of a word of the form $f_1 p(i, j) f_2$ into the word $f_1 p(j, i) f_2$. Two words f and g are said *homotopic* (denote $f \sim g$) if we can obtain one from the other by a sequence of elementary homotopies.

Remark If $f \sim g$ then $s_f = s_g$ and $l(f) = l(g)$.

Two galleries from $\mathcal{G}$ are called *homotopic* if their types are homotopic words. It follows from the remark above that two galleries from $\mathcal{G}$ which are homotopic must end at the same chamber.

Definition A word is called *reduced* if it is not homotopic to a word of the form $f_1 i i f_2$.

Proposition 1.4.1 *The word f is reduced if and only if the associated expression s_f is reduced (as element of W).*

PROOF It is not difficult to see that if s_f is not reduced, then there exist

$i, j \in \mathcal{I}_k$, such that $s_f = f_1 p(i, j) p(j, i) f_2$, if $m_{ij} = 3$ or $s_f = f_1 p(i, j) p(i, j) f_2$ if $m_{ij} \in \{2, 4, 6\}$.

In both cases we can find an elementary homotopy, namely given by $p(i, j) \mapsto p(j, i)$, which maps f to something of the form $g = g_1 i i g_2$. Conversely, if $f \sim g$, with g of the same form as above, then $s_f = s_{g_1} (s_i)^2 s_{g_2}$, which is, of course, a nonreduced expression.

□

Theorem 1.4.2 *a) The gallery of type f from $\mathcal{G}$ is minimal iff f is a reduced word.*

b) For two reduced words f and g , $s_f = s_g$ iff $f \sim g$.

PROOF (see [33], Th.1.4.)

□

As an immediate consequence one obtains:

Corollary 1.4.3 *Any two minimal galleries which end at the same chamber are homotopic.*

Thus, in terms of our previous identification, the set $\mathcal{G}_m$ of minimal galleries from $\mathcal{G}$ is just the set of reduced words. A very important object for our future developments is also the set $\mathcal{G}_m / \sim = \{[f] : f \in \mathcal{G}_m\}$ of all homotopy classes of minimal galleries. As final conclusion we can state now:

Corollary 1.4.4 *The correspondance $[f] \mapsto s_f$ is well defined and establishes a bijection between the sets $\mathcal{G}_m / \sim$ and W .*

1.5 Homology of M via homotopy of galleries

The purpose declared at the end of paragraphe 3 was to elucidate the ambiguities which can occur in constructing the HPT-cycles. So let us return to the context temporarily abandoned, i.e. to our compact, connected, complete and full isoparametric submanifold $X : M^n \rightarrow \mathbb{R}^{n+k}$. Take $x_0 \in M$ arbitrarily and $a \in \Delta(x_0)$. We already noticed that f_a is a Morse function whose critical set is $C(f_a) = W.x_0$. Fix x such a critical point and recall that in order to construct the HPT-cycle we must first consider the intersections of $[ax]$ with the focal hyperplanes from $\nu_{x_0}(M)$ (if necessary, we move a inside

$\Delta(x_0)$ such that $[ax]$ doesn't pass through the origin). Ordered from a to x , let this hyperplanes be $L_1, L_2, \dots, L_r$. By the recursive rule: x_{j+1} is obtained by reflecting x_j in L_j , $0 \leq j \leq r-1$, we get the sequence of points $x_1, x_2, \dots, x_{r-1}$, $x_r = x$ of M . Each L_j becomes in this way the common wall of the chambers $\Delta(x_j)$ and $\Delta(x_{j+1})$, $1 \leq j \leq r-1$, so we can consider in this way the sequence $i_1, i_2, \dots, i_r$ of fundamental indices determined by:

$$L_1 = l_{i_1}(x_0) = l_{i_1}(x_1), \dots, L_r = l_{i_r}(x_{r-1}) = l_{i_r}(x_r). \quad (1.7)$$

In the language of our previous section, we obtained a gallery (which turns out later to be minimal), joining $\Delta(x_0)$ to $\Delta(x)$, namely this gallery whose type is $i_1 i_2 \dots i_r$.

Let now $\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_r}$ be the involutive automorphisms of M induced respectively by the curvature spheres $S_{i_1}, S_{i_2}, \dots, S_{i_r}$. Notice that for each $1 \leq j \leq r$ holds:

$$x_j = \varphi_{i_j} \dots \varphi_{i_1} x_0.$$

Recall that the antipodal maps considered above are in fact end-point maps of the form:

$$\varphi_j(x) = \pi_{u_j}(x) = x + u_j(x), \quad x \in M,$$

where $u_j(x) = \frac{2v_j(x)}{\|v_j\|^2}$, $x \in M$, are parallel normal fields which do not focalize M , $1 \leq j \leq r$.

Lemma 1.5.1 *If $i, j \in \{1, \dots, p\}$, then φ_j maps the sphere $S_i(x)$ diffeomorphically onto the curvature sphere parallel to $S_i(x)$ which passes through $\varphi_j(x)$.*

PROOF At any point $y \in M$ the differential $(d\varphi_j)_y$ maps $E_i(y)$ isomorphically onto $E_{\sigma_j(i)}(\varphi_j(y))$, for a certain $\sigma_j(i) \in \{1, 2, \dots, p\}$ (cf. [32], Thm.6.3.2). It follows that φ_j maps the leaf $S_i(x)$ of E_i onto the leaf $S_{\sigma_j(i)}(\varphi_j(x))$ of $E_{\sigma_j(i)}$. On the other hand, $u_j|_{S_i(x)}$ is a parallel normal vector field on $S_i(x)$, so the image of the corresponding end-point map must be a sphere

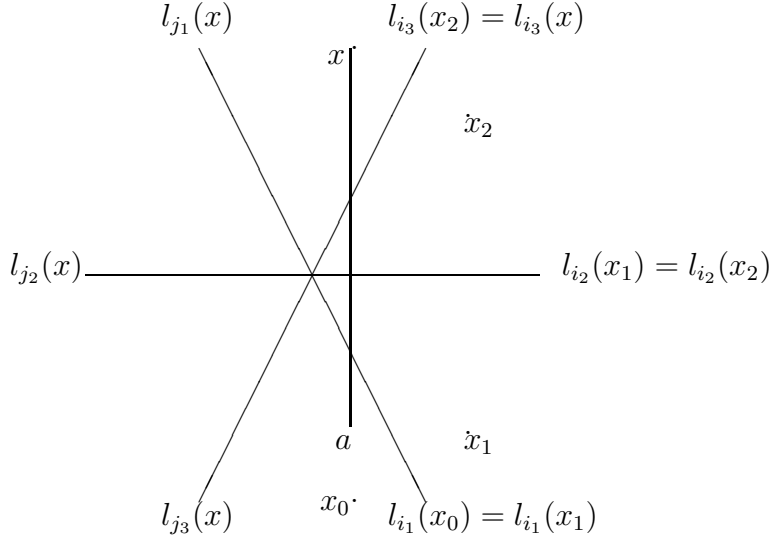


Figure 1.6

of the same dimension and the same focal set as $S_i(x)$, i.e. a sphere parallel to $S_i(x)$.

□

Regard now the hyperplanes $L_1, L_2, \dots, L_r$ from x , namely consider the set of indices $j_1, j_2, \dots, j_r \in \{1, 2, \dots, p\}$ (which are this time not any more fundamental), determined by:

$$L_1 = l_{j_1}(x), \dots, L_r = l_{j_r}(x) \quad (1.8)$$

and then take

$$N = \{(y_1, \dots, y_r) : y_1 \in S_{j_1}(x), y_2 \in S_{j_2}(y_1), \dots, y_r \in S_{j_r}(y_{r-1})\}$$

together with $u_N : N \rightarrow M$, $(y_1, \dots, y_r) \mapsto y_r$. The main result of section 1.3 says that (N, u_N) is a cycle of Bott-Samelson type for f_a at x and consequently $u_{N*}[N]$ is a linking cycle for f_a at x . The idea is now to compare

this latter cycle to those furnished by:

$$F = \{(z_1, \dots, z_r) : z_1 \in S_{i_1}(x_0), z_2 \in S_{i_2}(z_1), \dots, z_r \in S_{i_r}(z_{r-1})\}, \quad (1.9)$$

and $u_F : F \rightarrow M$, $(z_1, \dots, z_r) \mapsto z_r$. A first step will be made by:

Proposition 1.5.2 *Let z be an arbitrary point of M and $m \in \{1, \dots, r-1\}$.*

Then:

- (i) $l_{i_m}(z) = l_{j_m}(\varphi_{i_r}\varphi_{i_{r-1}}\dots\varphi_{i_{m+1}}z)$;
- (ii) *the map $\varphi_{i_r}\varphi_{i_{r-1}}\dots\varphi_{i_{m+1}}$ establishes a diffeomorphism between $S_{i_m}(z)$ and $S_{j_m}(\varphi_{i_r}\varphi_{i_{r-1}}\dots\varphi_{i_{m+1}}z)$;*
- (iii) $i_r = j_r$.

PROOF (i) Denote the element $\varphi_{i_r}\varphi_{i_{r-1}}\dots\varphi_{i_{m+1}}$ of W by φ and notice that $l_{i_m}(x_m) = l_{j_m}(\varphi x_m)$. In order to pass to an arbitrary point $z \in M$, it will be enough to consider the parallel transport in the normal bundle from $\nu_{x_m}(M)$ to $\nu_z(M)$ and use the fact that it commutes with φ .

(ii) The image of $S_{i_m}(z)$ through φ is, by the previous lemma, the curvature sphere through φz which focalizes in $l_{i_m}(z) = l_{j_m}(\varphi z)$, that is $S_{j_m}(\varphi z)$. All we have to do now in order to finish is to replace z by $\varphi_{i_m}(z)$.

(iii) By choosing $m = r-1$ in (i) we obtain $l_{i_r}(z) = l_{j_r}(\varphi_{i_r}z)$, or, if we replace z by $\varphi_{i_r}(z)$, $l_{i_r}(z) = l_{i_r}(\varphi_{i_r}z) = l_{j_r}(z)$.

□

Before stating the relationship between (N, u_N) and (F, u_F) in $H_*(M)$, we need some more information regarding in special the orientability in the case when $\mathcal{R} = \mathbb{Z}$ can be chosen.

Definition The F -cycle associated to an **arbitrary** word $f = i_1 i_2 \dots i_r$ is the pair (F_f, u_f) where

$$F_f = \{(z_1, z_2, \dots, z_r) : z_1 \in S_{i_1}(x_0), z_2 \in S_{i_2}(z_1), \dots, z_r \in S_{i_r}(z_{r-1})\}$$

and $u_f : F_f \rightarrow M$ is given by $(z_1, \dots, z_r) \mapsto z_r$, for any $(z_1, \dots, z_r) \in F_f$.

The manifold F_f defined above is an iterated bundle whose primordial total space and the successive fibers are curvature spheres of M . So, whenever both multiplicities are greater than 1, F_f is simply connected and orientable. An important role will be played by the fundamental homology class of such manifolds and, in the situation mentioned above, a natural orientation will

be needed. The following assumption will be from now on always present. It doesn't alterate the generality of the present context and will become concrete only in the next chapter:

Assumption In the case when both multiplicities are greater than 1, the orientation of all fundamental curvature spheres is fixed.

To orient curvature spheres is the same as to orient the corresponding curvature distributions. Since the tangent space to an arbitrary F_f at an arbitrary point $(z_1, \dots, z_r)$ is $E_{i_1}(z_1) \times \dots \times E_{i_r}(z_r)$ (where f is supposed to be $i_1 \dots i_r$), by the assumption above we have also oriented F_f . The following remark will play later an important rôle:

Remark 1 Suppose that both multiplicities of M are greater than 1. If $f = i_1 \dots i_r$ is an arbitrary word, take $f' = i_1 \dots i_{r-1}$ and consider the bundle map $\pi : F_f \rightarrow F_{f'}$, which associates to any $(z_1, \dots, z_r) \in F_f$ the element $(z_1, \dots, z_{r-1})$ of $F_{f'}$. The map $s : F_{f'} \rightarrow F_f$, $(z_1, \dots, z_{r-1}) \mapsto (z_1, \dots, z_{r-1}, z_{r-1})$, $(z_1, \dots, z_{r-1}) \in F_{f'}$ is a section of this bundle which embeds $F_{f'}$ in F_f . For any point $z = (z_1, \dots, z_{r-1}, z_{r-1}) \in F_{f'}$, the submanifold $F_{f'}$ and the fiber $(z_1, \dots, z_{r-1}) \times S_{i_r}(z_{r-1})$ and have in z a unique intersection, which is transversal of degree $+1$ in F_f .

Coming back to the context from above, we are now in position to prove:

Corollary 1.5.3 *The map $\Phi : F \rightarrow N$, $(z_1, z_2, \dots, z_r) \mapsto (\varphi_{i_r} \varphi_{i_{r-1}} \dots \varphi_{i_2} z_1, \dots, \varphi_{i_r} z_{r-1}, z_r)$ is a diffeomorphism which makes the following diagram commutative:*

$$\begin{array}{ccc}
 F & \xrightarrow{\Phi} & N \\
 & \searrow u_F & \swarrow u_N \\
 & M &
 \end{array}$$

Consequently, (F, u_F) is an another cycle of Bott-Samelson type at the point x , which induces in homology the cycle $u_{F*}[F] = \pm u_{N*}[N]$.

PROOF Since Φ is an automorphism of $M^{\times r}$, it will be enough to show that it maps F into N and its inverse maps N into F . Take $(z_1, z_2, \dots, z_r)$ an arbitrary point in F and use succesively the previous proposition as follows: $\varphi_{i_r} \dots \varphi_{i_2} z_1 \in \varphi_{i_r} \dots \varphi_{i_2} S_{i_1}(x_0) = S_{j_1}(\varphi_{i_r} \dots \varphi_{i_2} \varphi_{i_1} x_0) = S_{j_1}(x)$,

$$\varphi_{i_r} \dots \varphi_{i_3} z_2 \in \varphi_{i_r} \dots \varphi_{i_3} S_{i_2}(z_1) = S_{j_2}(\varphi_{i_r} \dots \varphi_{i_3} \varphi_{i_2} z_1),$$

.....
 $\varphi_{i_r} z_{r-1} \in \varphi_{i_r} S_{i_{r-1}}(z_{r-2}) = S_{j_{r-1}}(\varphi_{i_r} \varphi_{i_{r-1}} z_{r-2})$
and finally $z_r \in S_{i_r}(z_{r-1})$, hence $\varphi_{i_r} z_{r-1}$ and z_r are on the same S_{i_r} -sphere,
i.e. $z_r \in S_{i_r}(\varphi_{i_r} z_{r-1}) = S_{j_r}(\varphi_{i_r} z_{r-1})$.

In order to prove the second assertion of the Corollary, observe that the map $f_a \circ u_F = f_a \circ u_N \circ \Phi$ has a unique maximum, which is nondegenerate, at $\Phi^{-1}(x, x, \dots, x) = (x_1, x_2, \dots, x_r)$.

□

As a subsidiary remark, it follows from the considerations above:

Corollary 1.5.4 *We can join any two points of M by a piecewise smooth curve with at most p components, all of them being segments of great circles in fundamental curvature spheres.*

PROOF Consider an arbitrary F-cycle (F, u) which generates the top-homology of M , i.e. $u_*[F] = \pm[M]$, and everything follows from the surjectivity of u .

□

We consider now for two arbitrary fundamental indices i and j a normal vector $v \in \nu(M)_{z_0}$ with $z_0 + v(z_0)$ belonging to $l_i(z_0)$, $l_j(z_0)$ and no other focal hyperplane (z_0 is here an arbitrary point in M). Denote by $S_{ij}(z_0)$ the connected component of z_0 in $\pi_v^{-1}(\pi_v(z_0))$, which is a rank 2 isoparametric submanifold having $l_i(z_0) \cap \xi_v^{z_0}$ and $l_j(z_0) \cap \xi_v^{z_0}$ as focal lines through z_0 . The restrictions $E_i|_{S_{ij}(z_0)}$ and $E_j|_{S_{ij}(z_0)}$ are curvature distributions on $S_{ij}(z_0)$ and their leaves $S_i(z)$ respectively $S_j(z)$ through any $z \in S_{ij}(z_0)$ are still here curvature spheres.

Lemma 1.5.5 *Let us denote m_{ij} by m and consider z_m a point in M . The following statements are equivalent:*

- a) $z_m \in S_{ij}(z_0)$.
- b) *There exists a sequence of points $z_1, z_2, \dots, z_{m-1} \in M$ with $z_1 \in S_i(z_0)$, $z_2 \in S_j(z_1)$, $\dots$, $z_m \in S_{\alpha(i,j)}(z_{m-1})$, where $\alpha(i, j)$ equals i or j , whether m is odd, respectively even.*

PROOF From b) follows $\pi_v(z_0) = \pi_v(z_1) = \dots = \pi_v(z_m)$, hence a) holds. The converse follows from Corollary 1.5.4.

□

It is not clear for the moment how could F-cycles help in the direction enounced at the beginning of the section : the F-cycle which, by Cor.1.5.3, replace the original HPT-cycle, is still so arbitrary as the HPT-cycle was. The following two results will bring some more light in this problem:

Lemma 1.5.6 *Let M be a (not necessarily irreducible) isoparametric hypersurface in the sphere, with $2 \leq p \leq 4$ distinct principal curvatures. Let m_1 and m_2 denote, as usual, the only values that the multiplicities can take. If f and g are the two distinct words which furnish the two minimal galleries of maximal length in the Weyl group W , then $u_{f*}[F_f] = (-1)^{s(m_1, m_2, p)} u_{g*}[F_g]$, where:*

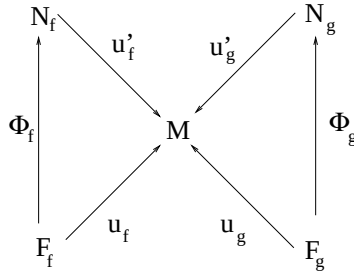
$$s(m_1, m_2, p) = \begin{cases} m_1 m_2, & \text{if } p = 2 \\ m_1 + m_2, & \text{if } p = 4 \\ 0, & \text{otherwise.} \end{cases}$$

PROOF Both terms of the identity we have to prove represent, up to a sign, the fundamental homology class of M . In order to see the sign which differentiate them, we make the following case-by-case analysis:

Case 1. $p = 2$. The isoparametric hypersurface M is simply the product of the spheres S^{m_1} and S^{m_2} . The two cycles we are referred in our statement are M itself and $S^{m_2} \times S^{m_1}$, the diffeomorphism which relates them having clearly degree $(-1)^{m_1 m_2}$.

Case 2. $p = 3$ and $m_1 = m_2 \geq 2$.

With the notations given by Figure 1.7, we have to compare the F-cycles (F_f, u_f) and (F_g, u_g) , given by $f = 121$, respectively $g = 212$. These come both from (different) HPT-cycles in the manner described in Cor.1.5.3, namely from (N_f, u'_f) , respectively (N_g, u'_g) .



The same corollary furnishes the diffeomorphisms $\Phi_f : F_f \rightarrow N_f$, $\Phi_g : F_g \rightarrow N_g$, which make the diagram below commutative. The concrete

form that these spaces have is:

$$F_f = \{(z_1, z_2, z_3) \in M^{\times 3} : z_1 \in S_1(x_0), z_2 \in S_2(z_1), z_3 \in S_1(z_2)\},$$

$$F_g = \{(z'_1, z'_2, z'_3) \in M^{\times 3} : z'_1 \in S_2(x_0), z'_2 \in S_1(z'_1), z'_3 \in S_2(z'_2)\},$$

$$N_f = \{(y_1, y_2, y_3) \in M^{\times 3} : y_1 \in S_2(x_3), y_2 \in S_3(y_1), y_3 \in S_1(y_2)\},$$

$$N_g = \{(y'_1, y'_2, y'_3) \in M^{\times 3} : y'_1 \in S_1(x_3), y'_2 \in S_3(y'_1), y'_3 \in S_2(y'_2)\}.$$

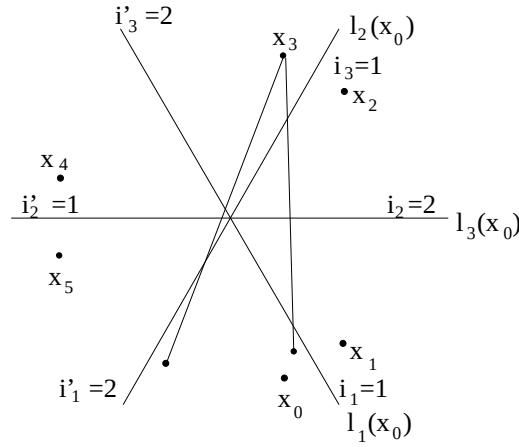


Figure 1.7

Also recall that $\Phi_f : F_f \rightarrow N_f$ is of the form $\Phi_f = (\varphi_1\varphi_2, \varphi_1, \text{id})$.

By the assumption above, the curvature spheres S_1 and S_2 are already oriented. One can see that S_3 is the diffeomorphic image of S_1 through φ_2 ; we orient S_3 in such a way, that this diffeomorphism has degree $+1$. On the other hand, S_3 is also the diffeomorphic image of S_2 through φ_1 . At this point we must anticipate partially the way in which the orientations of S_1 and S_2 will be chosen (see Thm. 2.1.2), namely that $\varphi_{1*}[S_2] = \varphi_{2*}[S_1]$, and conclude from here that also $\varphi_1|_{S_2} : S_2 \rightarrow S_3$ has degree $+1$. It can be easily seen now that Φ_f and Φ_g are both maps of degree $+1$. Hence we only have to see which is the difference between the two orientations induced on M by the HPT-cycles (N_f, u_f) , respectively (N_g, u_g) . To this end, we consider the differentials of u_f and u_g at (x_3, x_3, x_3) as obtained during the proof of Thm.1.3.2, and reduce our problem to compare the orientations of TM_{x_3} regarded as $E_2(x_3) \oplus E_3(x_3) \oplus E_1(x_3)$, respectively $E_1(x_3) \oplus E_3(x_3) \oplus E_2(x_3)$.

But as long the summands are even dimensional, the two orientations are the same.

Case 3. $p = 4$ and m_1, m_2 are both greater than 1.

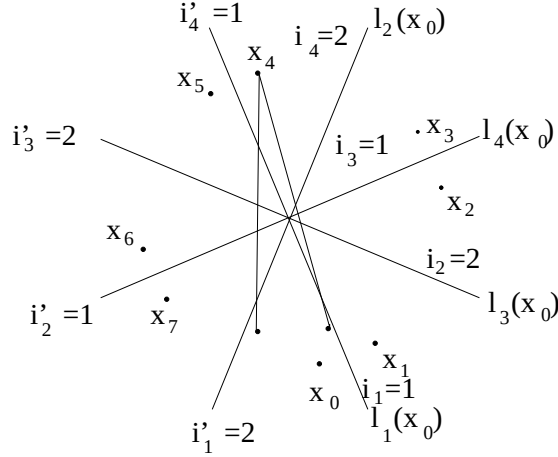


Figure 1.8

This time $f = 1212$, $g = 2121$,

$$F_f = \{(z_1, z_2, z_3, z_4) \in M^{\times 4} : z_1 \in S_1(x_0), z_2 \in S_2(z_1), z_3 \in S_1(z_2), z_4 \in S_2(z_3)\},$$

$$F_g = \{(z'_1, z'_2, z'_3, z'_4) \in M^{\times 4} : z'_1 \in S_2(x_0), z'_2 \in S_1(z'_1), z'_3 \in S_2(z'_2), z'_4 \in S_1(z'_3)\},$$

$$N_f = \{(y_1, y_2, y_3, y_4) \in M^{\times 4} : y_1 \in S_1(x_4), y_2 \in S_3(y_1), y_3 \in S_4(y_2), y_4 \in S_2(y_3)\},$$

$$N_g = \{(y'_1, y'_2, y'_3, y'_4) \in M^{\times 4} : y'_1 \in S_2(x_4), y'_2 \in S_4(y'_1), y'_3 \in S_3(y'_2), y'_4 \in S_1(y'_2)\}.$$

and $\Phi_f = (\varphi_2\varphi_1\varphi_2, \varphi_2\varphi_1, \varphi_2, \text{id})$.

We orient S_3 and S_4 in such a way, that $\varphi_1|_{S_2} : S_2 \rightarrow S_3$ and $\varphi_2|_{S_1} : S_1 \rightarrow S_4$ have degree +1. We must also know which is the degree of $\varphi_1|_{S_4} : S_4 \rightarrow S_4$. Once again we anticipate one of the results stated in Thm.2.1.2, namely

$$\begin{cases} \varphi_{1*}([S_2]) = [S_2] + d_{21}[S_1] \\ \varphi_{2*}([S_1]) = [S_1] + d_{12}[S_2] \end{cases}$$

where, up to a converse,

$$(d_{12}, d_{21}) = \begin{cases} (-1, -2), & \text{if } m_1 = m_2 = 2 \\ (0, 0), & \text{in all other cases.} \end{cases}$$

This means that $[S_4] = \varphi_{2*}([S_1])$ and $\varphi_{1*}([S_4]) = \varphi_{1*}\varphi_{2*}[S_1] = ((-1)^{m_1+1} + d_{12}d_{21})[S_1] + d_{12}[S_2]$. A simple reasoning shows now that $\varphi_{1*}[S_4] = (-1)^{m_1+1}[S_4]$. In a similar way, one obtains $\varphi_{2*}[S_3] = (-1)^{m_2+1}[S_3]$. Hence $\varphi_2\varphi_1\varphi_2$ maps S_1 onto S_1 diffeomorphically, changing the orientation by $(-1)^{m_1+1}$; $\varphi_2\varphi_1$ maps S_2 onto S_3 diffeomorphically, changing the orientation by $(-1)^{m_2+1}$. The degree of Φ_f is consequently $(-1)^{m_1+m_2}$. Of course, by symmetry, the same is the degree of Φ_g .

As in the previous situation, we only have to compare the orientations of TM_{x_4} given by its direct decompositions as $E_1(x_4) \oplus E_3(x_4) \oplus E_4(x_4) \oplus E_2(x_4)$, respectively $E_2(x_4) \oplus E_4(x_4) \oplus E_3(x_4) \oplus E_1(x_4)$. It is not difficult to see that the difference between these orientations is $(-1)^{m_1+m_2}$. □

Let us return now to the previous context when M is an arbitrary irreducible isoparametric submanifold of type different of G_2 , x_0 an arbitrary point of M and the Weyl group of M is regarded as the group generated by the reflections in the focal hyperplanes of $\nu_{x_0}(M)$. As fundamental Weyl chamber we choose, as usual, the chamber which contains x_0 . Consider now f and g the types of two homotopical galleries from $\mathcal{G}_m$. This means that g can be obtained from f by a sequence of alterations of the form $f_1^{(k)} p(i_k, j_k) f_2^{(k)} \mapsto f_1^{(k)} p(j_k, i_k) f_2^{(k)}$, $1 \leq k \leq t$, where $f_1^{(1)} p(i_1, j_1) f_2^{(1)} = f$, $f_1^{(t)} p(j_t, i_t) f_2^{(t)} = g$.

We associate to this homotopy of words the number $(-1)^{s(f,g)}$, where:

$$s(f, g) = \sum_{k=1}^t s(m_{i_k}, m_{j_k}, m_{i_k j_k}). \quad (1.10)$$

The following proposition will show that the number $(-1)^{s(f,g)}$ is well-defined, in the sense that it does not depend on the choice of the homotopy between f and g .

Proposition 1.5.7 *If M is not of type G_2 , then for any two homotopic words f and g holds:*

$$u_{f*}[F_f] = (-1)^{s(f,g)} u_{g*}[F_g].$$

PROOF It is, of course, enough to analyze the situation when $f = f_1 p(i, j) f_2$ and $g = f_1 p(j, i) f_2$, and to show that in this case holds $u_{f*}[F_f] =$

$(-1)^{s(m_i, m_j, m_{ij})} u_{g*}[F_g]$. For sake of simplicity consider $m_{ij} = 3$. The corresponding F-cycles are then:

$$F_f = \{(z_1, \dots, z_r) : \dots, z_t \in S_i(z_{t-1}), z_{t+1} \in S_j(z_t), z_{t+2} \in S_i(z_{t+1}), ***\},$$

$$F_g = \{(z_1, \dots, z_r) : \dots, z_t \in S_j(z_{t-1}), z_{t+1} \in S_i(z_t), z_{t+2} \in S_j(z_{t+1}), ***\}.$$

Consider

$$F_0 = \{(z_1, \dots, z_{t-1}, z_{t+2}, \dots, z_r) : \dots, z_{t+2} \in S_{ij}(z_{t-1}), ***\}$$

(the missing groups of relations "... " and "***" are easy to imagine and always the same) and also the map $p_f : F_f \rightarrow F_0$, $(z_1, \dots, z_r) \mapsto (z_1, \dots, \hat{z}_t, \hat{z}_{t+1}, \dots, z_r)$, which is well-defined and surjective, by Lemma 1.5.5. We will prove first that p_f is a map of degree ± 1 : to this end, we consider the sequence of points $x_1, \dots, x_r$ obtained from x_0 by applying successively the fundamental reflections dictated by f , in such a way that $(x_1, \dots, x_r)$ will be in F_f , and we show:

α) The preimage of $(x_1, \dots, x_{t-1}, x_{t+2}, \dots, x_r)$ through p_f reduces to $(x_1, \dots, x_r)$.

β) The map p_f is a local diffeomorphism at $(x_1, \dots, x_r)$.

Let us begin with α): We have to prove that $z_t \in S_i(x_{t-1}), z_{t+1} \in S_j(z_t), x_{t+2} \in S_i(z_{t+1})$ imply $z_t = x_t$ and $z_{t+1} = x_{t+1}$. In the isoparametric submanifold $S_{ij}(x_{t-1})$ take $a' \in \Delta(x_{t-1})$ such that the line segment $[a'x_{t+2}]$ intersects the only focal lines $l_i(x_{t-1}) \cap \xi_v^{x_{t-1}}$, $l_j(x_t) \cap \xi_v^{x_{t-1}}$ and $l_i(x_{t+1}) \cap \xi_v^{x_{t-1}}$, counted from a' to x_{t+2} . The critical point x_{t+2} of the distance function $g_{a'} : S_{ij}(x_{t-1}) \rightarrow \mathbb{R}$ admits as cycle of Bott-Samelson type the substitute of the HPT-cycle given by Cor.1.5.3, namely $\{(y_t, y_{t+1}, y_{t+2}) : y_t \in S_i(x_{t-1}), y_{t+1} \in S_j(y_t), y_{t+2} \in S_i(y_{t+1})\}$ together with $u : (y_t, y_{t+1}, y_{t+2}) \mapsto y_{t+2}$. This means that $g_{a'} \circ u$ has at (x_t, x_{t+1}, x_{t+2}) an absolute maximum, hence from $u(z_t, z_{t+1}, x_{t+2}) = u(x_t, x_{t+1}, x_{t+2})$ follow the desired identities.

β) Distinguish in F_f the following three submanifolds:

$$F_f^1 = \{(z_1, \dots, z_{t-1}, \varphi_{i_t} z_{t-1}, \dots, \varphi_{i_r} \dots \varphi_{i_t} z_{t-1}) : \dots\},$$

$$F_f^2 = \{(x_1, \dots, x_{t-1}, z_t, z_{t+1}, z_{t+2}, \varphi_{i_{t+3}} z_{t+2}, \varphi_{i_r} \dots \varphi_{i_{t+3}} z_{t+2}) : z_t \in S_i(x_{t-1}), z_{t+1} \in S_j(z_t), z_{t+2} \in S_i(z_{t+1})\},$$

$$F_f^3 = \{(x_1, \dots, x_{t-1}, x_t, x_{t+1}, x_{t+2}, z_{t+3}, \dots, z_r) : ***\}.$$

Similarly, consider in F_0 :

$$F_0^1 = \{(z_1, \dots, z_{t-1}, \varphi_{i_{t+2}} \varphi_{i_{t+1}} \varphi_{i_t} z_{t-1}, \dots, \varphi_{i_r} \dots \varphi_{i_t} z_{t-1}) : \dots\},$$

$$F_0^2 = \{(x_1, \dots, x_{t-1}, z_{t+2}, \varphi_{i_{t+3}} z_{t+2}, \dots, \varphi_{i_r} \dots \varphi_{i_{t+3}} z_{t+2}) : z_{t+2} \in S_{ij}(x_{t-1})\},$$

$$F_0^3 = \{(x_1, \dots, x_{t-1}, x_{t+2}, z_{t+3}, \dots, z_r) : * * *\}.$$

Then

$$(TF_f)_x = (TF_f^1)_x \oplus (TF_f^2)_x \oplus (TF_f^3)_x, \quad (TF_0)_x = (TF_0^1)_{\hat{x}} \oplus (TF_0^2)_{\hat{x}} \oplus (TF_0^3)_{\hat{x}},$$

where $x = (x_1, x_2, \dots, x_r)$ and $\hat{x} = p_f(x) = (x_1, \dots, \hat{x}_t, \hat{x}_{t+1}, \dots, x_r)$. Also notice that $p_f(F_f^\gamma) \subset F_0^\gamma$, $\gamma \in \{1, 2, 3\}$ and moreover, the restrictions $p_f|_{F_f^1} : F_f^1 \rightarrow F_0^1$ and $p_f|_{F_f^3} : F_f^3 \rightarrow F_0^3$ are essentially identical maps. As about $p_f|_{F_f^2}$, it reduces essentially to the F-cycle associated to $g_{a'}$ at x_{t+2} and is consequently a local diffeomorphism at x .

In a similar way we can prove that also $p_g : F_g \rightarrow F_0$ is a map of degree ± 1 . From the commutativity of the diagram below we deduce $u_{f*}[F_f] = u_{0*}p_{f*}[F_f] = \pm u_{0*}[F_0] = \pm u_{g*}[F_g]$.

$$\begin{array}{ccccc}
& & p_f & & \\
& & \longrightarrow & & \\
F_f & & & F_0 & \longleftarrow & F_g \\
& & & \downarrow u_0 & & \\
& & u_f & & u_g & \\
& & & M & &
\end{array}$$

But it is easy to see in the reasoning above that the sign which differentiate the two cycles above is the same as the sign which differentiate the F-cycles in $S_{ij}(x_{t-1})$ corresponding to the two different choices of minimal galleries of maximal length from x_{t-1} to x_{t+2} , which are respectively associated to the reduced words $p(i, j) = iji$ and $p(j, i) = jij$.

□

Remark Also in the case G_2 holds $u_{f*}[F_f] = (-1)^{s(f,g)} u_{g*}[F_g]$, where f and g are the two minimal galleries of maximal length in the corresponding Weyl group. By reasons that will become clear only later, we do not need the exact value of $s(f, g)$ in this case (see Cor.2.3.9).

Another important step will be made by:

Proposition 1.5.8 *If the word f is not reduced (i.e. the associated gallery is not minimal), then $u_{f*}[F_f] = 0$.*

PROOF Because of the previous proposition it will be enough to consider $f = f_1 i i f_2$, so F_f will be of the form $\{(z_1, \dots, z_r) : \dots, z_t \in S_i(z_{t-1}), z_{t+1} \in S_i(z_t), \dots\}$. Take $F' = \{(z_1, \dots, \hat{z}_t, \dots, z_r) : \dots, z_{t+1} \in S_i(z_{t-1}), \dots\}$ and $u' : F' \rightarrow M$, $(z_1, \dots, \hat{z}_t, \dots, z_r) \mapsto z_r$. If $p : F \rightarrow F'$ is defined by the rule $(z_1, \dots, z_r) \mapsto (z_1, \dots, \hat{z}_t, \dots, z_r)$, $(z_1, \dots, z_r) \in F$, then evidently $u' \circ p = u_f$. It follows $u_{f*}[F_f] = u'_{*}p_{*}[F_f] = 0$, simply because $\dim F_f > \dim F'$.

□

The essence of the section is concentrated in the following theorem:

Theorem 1.5.9 *Fix in $\mathcal{G}_m$ a certain subset $\mathcal{G}_m^0$ which consists of representatives of the quotient space $\mathcal{G}_m / \sim$. If $[\mathcal{G}_m^0]$ and $[W]$ are the free $\mathcal{R}$ -modules generated by the elements of $\mathcal{G}_m^0$ respectively $[W]$, then the correspondances*

$$f \mapsto s_f,$$

respectively

$$f \mapsto u_{f*}[F_f],$$

are well-defined and can be extended to isomorphisms of $\mathcal{R}$ -modules between $[W]$, $[\mathcal{G}_m^0]$ and $H_(M, \mathcal{R})$.*

PROOF There is nothing more to be proved about the first map (see Cor. 1.4.4).

The second map is well-defined up to a sign, by Prop. 1.5.7. On the other hand, for any $f \in \mathcal{G}_m$, $u_{f*}[F_f]$ is a linking cycle at $s_f(x_0)$: One knows that there exists a gallery of a certain type $g \in \mathcal{G}_m$, joining $\Delta(x_0)$ with $\Delta(s_f(x_0))$, such that $u_{g*}[F_g]$ is a linking cycle at $s_f(x_0)$ (see Cor. 1.5.3). Since $u_{g*}[F_g] \neq 0$, it follows from Prop. 1.5.7 that g describes a minimal gallery. But f and g being minimal, both ending at the same chamber, we have $f \sim g$ (see Cor. 1.4.3), hence $u_{f*}[F_f] = \pm u_{g*}[F_g]$.

We conclude by applying Thm. 1.3.1, Thm. 1.3.2 and Cor. 1.4.4.

□

Remark 2. As we will see later, it will be often convenient to choose, even in an arbitrary way, a set $\mathcal{G}_m^0$ of representatives of $\mathcal{G}_m / \sim$ (equivalently, to fix a reduced expression for any element $w \in W$, or to fix a minimal gallery from x_0 to any wx_0 , $w \in W$) and then to identify $H_*(M, \mathcal{R})$ to one of the $\mathcal{R}$ -modules $[W]$ or $[\mathcal{G}_m^0]$ (the induced graduation of the latter one is given by $\deg(f) = \sum_{j=1}^r m_{i_j}$, for any $f = i_1 i_2 \dots i_r \in \mathcal{G}_m^0$).

1.6 The homology of M_v

We have seen in the third section that isoparametric submanifolds are embedded in the ambient Euclidean space in a very reasonable manner, which offer the possibility to obtain their homology module only from the "infinitesimal" data (Weyl group and multiplicities): the basic tool was the Morse theory of nondegenerate distance functions, namely the fact that they are $\mathcal{R}$ -perfect. The tautness property is stated in [19] also for focal manifolds, but no direct proof is done there. We will try in the sequel to assume ourselves this task : the main goal will be also here the construction of cycles of Bott-Samelson type, that will be obtained in a natural way by "projecting" the previous HPT-cycles from M to M_v .

As usual, M^n will be a compact, connected, complete and full isoparametric submanifold in $\mathbb{R}^{n+k}$. Consider now v a normal parallel field and associate to it all objects defined in the first half of section 2 (see Thm.1.2.2). We begin with the following technical result:

Lemma 1.6.1 *For any $x \in M$ and any $w \in W$,*

$$\pi_v(w.x) = w.\pi_v(x).$$

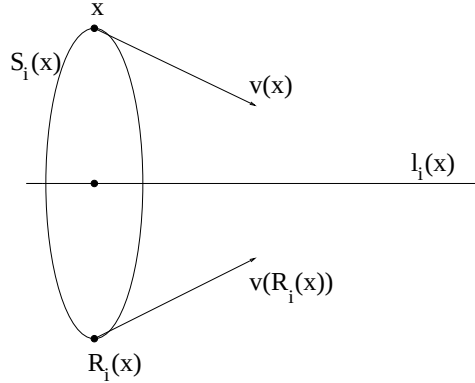


Figure 1.9

PROOF It will be enough to consider the case $w = R_i^x$: because v is parallel along the sphere which focalizes in $l_i(x)$, it follows that its value

$R_i^x(x) + v(R_i^x(x))$ at the antipodal point of x must be the reflection of $x + v(x)$ (also see Figure 1.9).

□

The following result describes completely the critical set of a distance function on M_v :

Theorem 1.6.2 (see [19]) *Let x_0 be an arbitrary point in M and $a \in \nu_{x_0}(M)$ be W -regular. Denote by g_a the associated distance function on M_v and take $x_0^* = \pi_v(x_0) \in M_v$. Then:*

(i) $C(g_a) = \pi_v(W.x_0) = W.x_0^*$, being constituted only of nondegenerate critical points;

(ii) if $x \in W.x_0$ and x^* denotes $\pi_v(x)$, then the negative space of g_a at x^* is

$$\sum \{E_j : j \notin J(v) \text{ and the line segment } [ax^*] \text{ meets } l_j(x)\}.$$

Consequently, the index of x^* as critical point of g_a equals the number of focal hyperplanes $l_j(x)$, $j \in \{1, 2, \dots, p\} \setminus J(v)$ intersected by the line segment $[ax^*]$ (equivalently, the focal hyperplanes $l_j(x)$, $j \in \{1, 2, \dots, p\}$, intersected by the line segment $[ax^*]$), each counted with its multiplicity.

We will try now to obtain homological information about M_v from the Morse function g_a . In order to do that, we need the following general result of the theory of Weyl groups (see for instance [20], section 1.10).

Lemma 1.6.3 *One can find a system of representatives W^v of the coset space W/W_v with the property that for any $u \in W^v$ and any $w \in uW_v$ there exists a unique element $u' \in W_v$ with $w = uu'$ and $l(w) = l(u) + l(u')$. Particularly, u is the unique element of its coset uW_v having minimal length.*

Following [20] we will call the elements of W^v the *minimal coset representatives* of W/W_v .

In the context of Thm.1.6.2, W_v is the stabilizer of x_0^* , hence the critical set of g_a is $W^v.x_0^*$, in a natural bijective correspondance with W^v . If f_a denotes the distance function from a on M , then its critical set $C(f_a) = W.x_0$ can be partitioned as $W.x_0 = \bigsqcup \{uW_v.x_0 : u \in W^v\}$, an element $uW_v.x_0$ of the partition consisting exactly of these points of $C(f_a)$ which project through π_v on $u.x_0^*$.

Fix now $u \in W^v$ a minimal representative, take the corresponding critical point ux_0 of f_a , together with its corresponding HPT-cycle $(N(ux_0), p_u)$. It

is maybe not surprising that a cycle of Bott-Samelson type for g_a at ux_0^* is obtained simply from the previous cycle after composing with π_v :

Proposition 1.6.4 *The index of any point of $C(f_a)$ which projects to ux_0^* is at least equal to $\text{index}(g_a, ux_0^*)$. The only point of $uW_v x_0$ of index equal to $\text{index}(g_a, ux_0^*)$ is ux_0 . The pair $(N(ux_0), \pi_v \circ p_u)$ is a cycle of Bott-Samelson type for g_a at ux_0^* .*

The proof of Prop.1.6.4 needs some preliminary results. Only by technical reasons we suppose that a lies in the interior of $\Delta(x_0)$.

Lemma 1.6.5 *Consider $w \in W$ and s_i a fundamental reflection with $l(ws_i) > l(w)$. Then $\text{index}(f_a, ws_i x_0) > \text{index}(f_a, wx_0)$ and $f_a(ws_i x_0) > f_a(wx_0)$. Consequently, for any $u \in W^v$ and any $x \in uW_v x_0 \setminus \{ux_0\}$, holds $\text{index}(f_a, ux_0) < \text{index}(f_a, x)$ and $f_a(ux_0) < f_a(x)$.*

PROOF Since $l(ws_i) = l(w) + 1$, it follows that the chambers $\Delta(x_0)$ and $\Delta(ws_i x_0)$ are separated by the same focal hyperplanes which separate $\Delta(x_0)$ and $\Delta(wx_0)$ and additionally $l_i(wx_0)$. In particular, the segment $[a, wx_0]$ does not cross $l_i(wx_0)$. All inequalities are now clear. □

Lemma 1.6.6 *If $u \in W^v$, then the line segment $[a, ux_0]$ cannot cross hyperplanes of the form $l_j(ux_0)$ with $j \in J(v)$.*

PROOF Let us suppose that something like that happens: reflect ux_0 in $l_j(ux_0)$ and obtain $x = \varphi_j(ux_0)$. The following diagram is commutative,

$$\begin{array}{ccc} M & \xrightarrow{\varphi_j} & M \\ & \searrow \pi_v \quad \swarrow \pi_v & \\ & M_v & \end{array}$$

hence $\pi_v \varphi_j(ux_0) = \pi_v(ux_0)$, consequently $\varphi_j(ux_0) \in uW_v x_0$. On the other hand, $d(a, ux_0) > d(a, x)$, in contradiction with Lemma 1.6.5. □

Lemma 1.6.7 *An element w of W is a minimal representative (i.e. it belongs to W^v) if and only if the line segments $[a, wx_0]$ and $[a, wx_0^*]$ intersect the same set of focal hyperplanes.*

PROOF Because of Lemma 1.6.5, we only have to prove that if u is a minimal representative, then the segments $[a, ux_0]$ and $[a, ux_0^*]$ cross the same focal hyperplanes. One already knows that this is true for $[a, ux_0]$ and $[a, ux_0^*]$, since ux_0 and ux_0^* are in the same chamber. If a certain $l_j(ux_0)$ would cross $[a, ux_0]$ but not $[a, ux_0^*]$, it would follow $ux_0^* \in l_j(ux_0)$, i.e. $j \in J(v)$, in contradiction with the previous lemma.

□

PROOF OF PROPOSITION 1.6.4 The only fact it remained to be proved is that the pair $(N(ux_0), \pi_v \circ p_u)$ is a cycle of Bott-Samelson type for g_a at ux_0^* . The proof that we will give in the sequel is inspired from the proof of Thm.1.3.2. Also now the line segment $[a, ux_0]$ meets successively the only focal hyperplanes $l_{j_1}(ux_0), \dots, l_{j_r}(ux_0)$, counted from a to ux_0 , and the same can be said about the segment $[a, ux_0^*]$. Denoting $N(ux_0)$ simply by N , all we have to prove is:

- (i) $y_0 = (ux_0, \dots, ux_0)$ is the unique maximum point of $g_a \circ \pi_v \circ p_u$.
- (ii) $d(\pi_v \circ p_u)_{y_0}$ maps TN_{y_0} isomorphically onto the negative space of g_a at ux_0^* .
- (iii) if m_{j_i} are all greater than 1, then N is orientable.

The point (ii) follows easily from the fact that no j_i belongs to $J(v)$. We take now $y = (y_1, y_2, \dots, y_r) \in N \setminus \{y_0\}$ and prove that $g_a \circ \pi_v(y_r) < g_a \circ \pi_v(ux_0)$, i.e. $d(a, \pi_v(y_r)) < d(a, ux_0^*)$. The main idea of the proof can be easily understood from the particular situation $r = 2$, $j_1 = 1$, $j_2 = 2$ (see Figure 1.10).

Consider the vector $\tilde{v}(ux_0) = a - ux_0^* \in \nu(M)_{x_0}$ and extend it to a normal parallel field $\tilde{v}$ along M . Let z_1 and z_2 be the points where the line segment $[a, ux_0^*]$ meets $l_1(ux_0)$ respectively $l_2(ux_0)$. At the point $y_1 \in S_1(ux_0)$ take the vectors $v(y_1)$ and $\tilde{v}(y_1)$ obtained by parallel transport along the sphere $S_1(ux_0)$ which focalizes in $l_1(ux_0)$: this means that if $a_1 = \pi_v(y_1) + \tilde{v}(y_1)$, then the line segment $[a_1, \pi_v(y_1)]$ crosses first of all the hyperplane $l_1(ux_0) = l_1(y_1)$ at the point $z_1^{(1)} = z_1$ and then $l_2(y_1)$ at $z_2^{(1)}$. A similar reasoning is available when passing from y_1 to y_2 along $S_2(y_1)$: if $a_2 = \pi_v(y_2) + \tilde{v}(y_2)$, then the line segment $[a_2, \pi_v(y_2)]$ crosses $l_1(y_2)$ in $z_1^{(2)}$ and then $l_2(y_2)$ in $z_2^{(2)} = z_2^{(1)}$.



11

38

now what can be said about the cycles associated to the other critical points of f_a which also project to ux_0^* . The answer is very simple: they are mapped to 0.

Theorem 1.6.9 *Among the critical points of f_a , the only ones whose HPT-cycles are not vanished through π_{v*} are those who correspond to minimal representatives of W/W_v . The HPT-cycles of the latter ones are mapped to linking cycles of g_a .*

PROOF If u is a minimal representative, we associate to an arbitrary $x = wx_0 \in uW_vx_0 \setminus \{ux_0\}$ the HPT-cycle $(N(x), p)$. By Lemma 1.6.3 there exists $u' \in W_v$, $u' \neq 1$ with $w = uu'$ and $l(w) = l(u) + l(u')$. One can associate to u and u' the reduced expressions $u = s_f$ and $u' = s_{f'}$, where f and f' are two words in $\mathcal{I}_k$, actually f' word in $I(v)$. Consequently $w = s_f s_{f'}$ is also a reduced expression. Replace the HPT-cycle $p_*[N(x)]$ by the F-cycle associated to the reduced expression of w above (see Cor 1.5.3 and Thm.1.5.8), of the form

$$F_{ff'} = \{(z_1, \dots, z_q) : z_1 \in S_{j_1}(x_0), \dots, z_{q-1} \in S_{j_{q-1}}(z_{q-2}), z_q \in S_{i_q}(z_{q-1})\},$$

together with $p_{ff'} : F_{ff'} \rightarrow M$, $(z_1, \dots, z_q) \mapsto z_q$, $(z_1, \dots, z_q) \in F_{ff'}$. As letter of the word f' , i_q belongs to $I(v)$. Consider now $F = \{(z_1, \dots, z_{q-1}) : z_1 \in S_{j_1}(x_0), \dots, z_{q-1} \in S_{j_{q-1}}(z_{q-2})\}$ and $p' : F \rightarrow M$, which maps an arbitrary $(z_1, \dots, z_{q-1}) \in F$ to z_{q-1} . Since π_v focalizes any S_{i_q} -sphere, the following diagram is commutative:

$$\begin{array}{ccc} F_{ff'} & \xrightarrow{h} & F \\ & \searrow \pi_v \circ p_{ff'} & \swarrow \pi_v \circ p' \\ & M_v & \end{array}$$

where $h(z_1, \dots, z_q) = (z_1, \dots, z_{q-1})$, for any $(z_1, \dots, z_q) \in F_{ff'}$. But $\dim F < \dim F_{ff'}$, hence $h_*[F_{ff'}] = 0$, and so $(\pi_v \circ p_{ff'})_*[F_{ff'}] = 0$ or, equivalently, $\pi_{v*}p_*[N(x)] = 0$.

□

1.7 Schubert cycles on Grassmannians as cycles of Bott-Samelson type

Recall that for any $1 \leq n \leq N$, the Grassmanian $G_n(\mathbb{R}^N)$ is the set of all n -planes (i.e. n -dimensional vector subspaces) of $\mathbb{R}^N$. Equipped with a natural topology and a natural differentiable structure (see for instance [12]), it represents an important mathematical object, whose study is essential in the developement of various mathematical theories: characteristic classes, symmetric spaces (intrinsic or extrinsic), isoparametric submanifolds etc. We are here especially interested in this latter approach: the general considerations made in the second half of section 2 (see Thm.1.2.3) can be applied to the symmetric space $SL(n, \mathbb{R})/SO(n)$. As we already noticed at the end of the same section (see Remark 1.), a similar treatment is available for the dual space (which is of compact type) $SU(N)/SO(N)$. More precisely, the last space is furnished by the complex conjugation automorphism σ of $SU(N)$, $\sigma(a) = \bar{a}$, for any $a \in SU(N)$. The corresponding decomposition of the Lie algebra $\mathfrak{su}(N) = \{x \in \mathbb{R}^{N \times N} : x \text{ is skew-hermitic of trace } 0\}$ is

$$\mathfrak{su}(N) = \mathfrak{o}(N) \oplus iS^0(N),$$

where $S^0(N)$ is the space of symmetric real $N \times N$ matrices of trace 0 and $\mathfrak{o}(N)$ the Lie algebra of $SO(N)$, i.e. the space of skew-symmetric real $N \times N$ matrices. As a maximal abelian subspace of $iS^0(N)$ we choose $i\mathfrak{a}^0$, where $\mathfrak{a}^0$ denotes the space of diagonal real matrices of trace 0. The corresponding isotropy representation is, up to a multiplication by i , as follows :

$$SO(N) \times S^0(N) \rightarrow S^0(N), (a, x) \mapsto axa^{-1},$$

for any $a \in SO(N)$ and any $x \in S^0(N)$. Recall that the roots of the symmetric space are, also up to a multiplication by i , of the form $\alpha_{ij} : \mathfrak{a}^0 \rightarrow \mathbb{R}$, $Diag(\lambda_1, \dots, \lambda_N) \mapsto \lambda_j - \lambda_i$, where $1 \leq i \neq j \leq N$ are two arbitrary indices. Since $\mathfrak{a}^0$ is maximal abelian also in $\mathfrak{su}(N)$, any of the roots above has multiplicity equal to 1. An element $x \in \mathfrak{a}^0$ is regular iff its entries are any two different; the corresponding orbit $SO(N).x$ is in this case a full isoparametric submanifold in $S^0(N)$, the latter one endowed with the Euclidean metric given by:

$$\langle x, y \rangle = \text{tr } xy,$$

for any two $x, y \in S^0(N)$.

From our point of view, it will be more convenient to have to do with the space $U(N)/O(N)$ rather than $SU(N)/SO(N)$. We will try in the sequel to explain why studying isotropy orbits of $SU(N)/SO(N)$ is the same as studying isotropy orbits of $U(N)/O(N)$. The involutive automorphism of $U(N)$ corresponding to the last symmetric space is also given by the complex conjugation and at the level of Lie algebras we have the eigenspace decomposition

$$\mathfrak{u}(N) = \mathfrak{o}(N) \oplus iS(N),$$

where $\mathfrak{u}(N)$ is the space of all skew-hermitic complex matrices and $S(N)$ the space of all real symmetric matrices. A maximal abelian subalgebra of $S(N)$ is the space $\mathfrak{a}$ of all diagonal $N \times N$ matrices; the roots are nothing but extensions of the previous α_{ij} to $\mathfrak{a}$, $1 \leq i \neq j \leq N$. The isotropy representation is still given by conjugation, as follows:

$$O(N) \times S(N) \rightarrow S(N), (a, x) \mapsto axa^{-1},$$

with $a \in O(N)$ and $x \in S(N)$. Let us now compare the orbital foliations of the two symmetric spaces from above:

$$S^0(N) = \bigsqcup \{SO(N).x_0 : x_0 \in \mathfrak{a}^0\},$$

repectively

$$S(N) = \bigsqcup \{O(N).x : x \in \mathfrak{a}\}.$$

It follws from a classical result of linear algebra that $SO(N).x = O(N).x$, for any $x \in \mathfrak{a}$. Moreover, notice that always holds $SO(N).x = SO(N).x_0 + \alpha I$, where $x_0 = x - \alpha I \in \mathfrak{a}^0$, i.e. $\alpha = \frac{1}{N} \text{tr} x$.

Consequently, from the point of view of the isotropy orbits, the only change which occurs if considering $U(N)/O(N)$ instead of $SU(N)/SO(N)$ is an additional unit in the dimension of the ambient space where the orbits are sitting: if the principal isotropy orbit of the (regular) element $x_0 \in \mathfrak{a}^0$ is regarded in $S(N)$ and not in $S^0(N)$, it will be not any more full, the curvature normals at x_0 are not any more able to generate the whole normal space at x_0 . But as long we are interested only about height functions and related facts, this will be no inconvenient: to any $a \in \mathfrak{a}$ we can associate $a_0 = a - \frac{1}{N}(\text{tra})I \in \mathfrak{a}^0$ in such a way that the corresponding height functions h_a and h_{a_0} differ only by a constant on any isotropy orbit (also notice that, as long our submanifolds in Euclidean spaces are allways embedded in a sphere with center in 0, using distance functions or height functions is the same).

We are especially interested about the symmetric space $U(N)/O(N)$ and the isotropy orbits of the elements $x_1 = \text{Diag}(1, 2, \dots, n, n+1, \dots, n+1)$ and $x_2 = \text{Diag}(\underbrace{1, \dots, 1}_{n \text{ times}}, 0, \dots, 0)$ of $\mathfrak{a}$. Their isotropy orbits are

$$F(n) = O(N)/\underbrace{O(1) \times \dots \times O(1)}_{n \text{ times}} \times O(N-n),$$

space we will call here the *partial flag manifold*, respectively the Grassmannian expressed as

$$G_n(\mathbb{R}^N) = O(N)/O(n) \times O(N-n).$$

Actually, there are some natural identifications which are involved here: notice first that, in general, the $O(N)$ -orbit of an arbitrary $x \in S(N)$ is the set of all matrices of $S(N)$ which have the same spectrum (i.e. the same eigenvalues with respectively the same multiplicities) as x . In this way, a matrix of the orbit $O(N).x_1$ is completely determined if we specify the eigenspaces corresponding to the eigenvalues $1, 2, \dots, n$, which are supposed to be two by two orthogonal lines in $\mathbb{R}^N$; in other words, we can interpret $F(n)$ as:

$$F(n) = \{(L_1, \dots, L_n) : L_i \text{ 1-dimensional subspace of } \mathbb{R}^N, \\ L_i \perp L_j, \text{ for any two different indices } i \text{ and } j\}$$

Similarly, any matrix from $O(N).x_2$ is uniquely determined by its eigenspaces, particularly by its 1-eigenspace, which can be any n -dimensional vector subspace of $\mathbb{R}^N$. An another way to describe the Grassmannian is:

$$G_n(\mathbb{R}^N) = \{x \in S(N) : x^2 = x \text{ and } \text{tr} x = n\}.$$

Both $F(n)$ and $G_n(\mathbb{R}^N)$ inherit a differentiable and also a Riemannian structure, as closed submanifolds of the Euclidean space $S(N)$.

In a very few words we recall in the sequel the classical construction of the Schubert cycles on $G_n(\mathbb{R}^N)$. We associate first to an arbitrary n -plane a from $\mathbb{R}^N$ the following increasing sequence of numbers: $d_1 = \dim(\mathbb{R} \cap a)$, $d_2 = \dim(\mathbb{R}^2 \cap a), \dots, d_N = \dim(\mathbb{R}^N \cap a) = n$. We also consider $s_i(a) = \min\{j \in \{1, \dots, N\} : \dim(\mathbb{R}^j \cap a) = i\}$ and obtain an another sequence $1 \leq s_1(a) < s_2(a) < \dots < s_n(a) \leq N$. By figuring this values in a diagram we obtain the so-called *graph* of a (see Figure 1.11).

Let conversely be $\sigma = (s_1, s_2, \dots, s_n)$ with $1 \leq s_1 < s_2 < \dots < s_n \leq N$ a so-called *Schubert symbol* and associate

$C_\sigma = \{a \in G_n(\mathbb{R}^N) : \dim(a \cap \mathbb{R}^{s_k}) = k \text{ and } \dim(a \cap \mathbb{R}^{s_k-1}) = k-1, \text{ for all } 1 \leq k \leq n\}.$

In other words, C_σ is the set of all n -planes of $\mathbb{R}^N$ which have the prescribed graph σ .

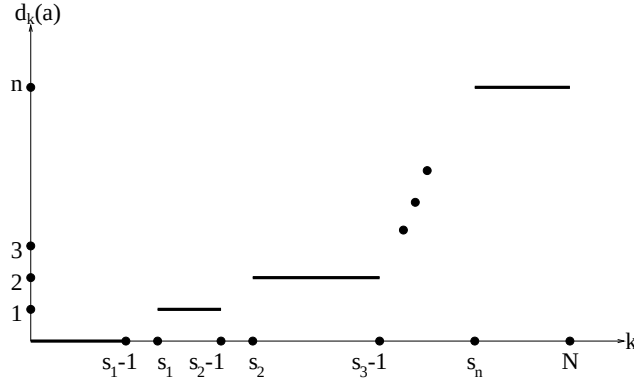


Figure 1.11: Schubert symbols as graphs of n -planes ($s_i = s_i(a)$ and $d_k(a)$ means $\dim(\mathbb{R}^k \cap a)$)

Consider also $Z_\sigma = \{a \in G_n(\mathbb{R}^N) : \dim(a \cap \mathbb{R}^{s_k}) \geq k, \text{ for all } 1 \leq k \leq n\}$, i.e. the set of all n -planes which grow faster as σ from the point of view of the graph. The point $e_\sigma = \text{span}\{e_{s_1}, e_{s_2}, \dots, e_{s_n}\} \in C_\sigma$ will play later an important role.

In a similar way we consider this time in $F(n)$ the subspace

$$\begin{aligned} \hat{Z}_\sigma &= \{(L_1, \dots, L_n) \in F(n) : L_k \subseteq \mathbb{R}^{s_k}, \text{ for all } 1 \leq k \leq n\} = \\ &= \{(L_1, \dots, L_n) \in F(n) : L_1 \in \mathbf{P}(\mathbb{R}^{s_1}), L_2 \in \mathbf{P}(\mathbb{R}^{s_2} \ominus L_1), \dots, \\ &L_n \in \mathbf{P}(\mathbb{R}^{s_n} \ominus (L_1 \oplus \dots \oplus L_{n-1}))\}. \end{aligned}$$

The idea is to substitute Z_σ by $\hat{Z}_\sigma$, in the intention to recuperate somehow the handicap of the first one of not admitting in general a differentiable structure. As about $\hat{Z}_\sigma$, it is the total space of an iterated bundle whose first base space and successive fibers are respectively the real projective spaces $\mathbf{P}(\mathbb{R}^{s_1})$, $\mathbf{P}(\mathbb{R}^{s_2-1})$, ..., $\mathbf{P}(\mathbb{R}^{s_n-(n-1)})$, hence it admits a natural structure of differentiable manifold of dimension $(s_1 - 1) + (s_2 - 2) + \dots + (s_n - n)$. The spaces Z_σ and $\hat{Z}_\sigma$ can be related through $f_\sigma : \hat{Z}_\sigma \rightarrow Z_\sigma$, which associates to an arbitrary $(L_1, L_2, \dots, L_n) \in \hat{Z}_\sigma$ the (direct) sum $L_1 + L_2 + \dots + L_n$. The map is well-defined, onto and invertible on C_σ : more precisely, if $a \in C_\sigma$, then $f_\sigma^{-1}(a) = (L_1, \dots, L_n)$, where the lines L_i are described by: $L_1 = a \cap \mathbb{R}^{s_1}$, $L_2 = (a \cap \mathbb{R}^{s_2}) \ominus L_1, \dots, L_n = (a \cap \mathbb{R}^{s_n}) \ominus (L_1 \oplus \dots \oplus L_{n-1})$.

Definition A *Schubert cycle* on $G_n(\mathbb{R}^N)$ is a pair of the form $(\hat{Z}_\sigma, f_\sigma)$, where σ is a Schubert symbol.

Another important related object we will need later is the space

$$\tilde{Z}_\sigma = \{(v_1, \dots, v_n) \in (\mathbb{R}^N)^{\times n} : \langle v_i, v_j \rangle = \delta_{ij} \text{ and } v_i \in \mathbb{R}^{s_i}, \text{ for all } 1 \leq i, j \leq n\},$$

which is clearly the total space of an iterated bundle whose first base space and successive fibers are respectively the spheres $S^{s_1-1}, S^{s_2-2}, \dots, S^{s_n-n}$. Notice for later use that the map $\tilde{Z}_\sigma \rightarrow \hat{Z}_\sigma$, $(v_1, v_2, \dots, v_n) \mapsto (\mathbb{R}v_1, \mathbb{R}v_2, \dots, \mathbb{R}v_n)$, $(v_1, v_2, \dots, v_n) \in \tilde{Z}_\sigma$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ -covering.

The main goal of the present section is to prove that if $x_0 = \text{Diag}(1, 2, \dots, N) \in \mathfrak{a}$ then the Schubert cycles $(\hat{Z}_\sigma, f_\sigma)$ defined above are actually cycles of Bott-Samelson type for the height function $h_{x_0} : G_n(\mathbb{R}^N) \rightarrow \mathbb{R}$, $x \mapsto \langle x_0, x \rangle$, $x \in G_n(\mathbb{R}^N)$. The first question concerns the nondegeneracy of h_{x_0} : that follows immediately from the regularity of x_0 in $\mathfrak{a}$. We must notice, by the way, that all reasonings made during this section are still available if x_0 is an arbitrary regular point in $\mathfrak{a}$, namely a diagonal matrix with two by two different entries; the relevance of the choice of x_0 as above will become clear only in our next section.

The normal space to $G_n(\mathbb{R}^N) = O(N).x_2$ at an arbitrary point a is :

$$\nu_a G_n(\mathbb{R}^N) = \{x \in S(N) : [x, a] = 0\}$$

(see for instance [5], Ch.II, (7.4)). In particular, a is a critical point of h_{x_0} iff $x_0 \in \nu_a G_n(\mathbb{R}^N)$, i.e. $[x_0, a] = 0$, hence $a \in \mathfrak{a}$. In other words, $C(h_{x_0}) = G_n(\mathbb{R}^N) \cap \mathfrak{a}$. One knows that the latter intersection is nothing but the W -orbit of x_2 (see for instance [18], Prop.2.2, Ch.VII), so the set of all diagonal matrices obtained by permuting the entries of x_2 . In terms of the identification from above, one can express that as follows:

$$C(h_{x_0}) = \{e_\sigma : \sigma \text{ is a Schubert cycle}\}.$$

Particularly, the number of these critical points equals $\binom{N}{n}$.

Lemma 1.7.1 *The restriction $h_{x_0}|_{Z_\sigma}$ has a unique maximum at e_σ .*

PROOF We start with the following general remark: if a, b are two planes in $\mathbb{R}^N$ (as usual, identified to the orthogonal projections on them, as elements of

$S(N)$), then $0 \leq \langle a, b \rangle \leq \dim a$; equality in the left hand side can be realized iff $a \perp b$ and in the right hand side iff $a \subseteq b$. In order to prove that, choose $(a_i)_{1 \leq i \leq l}$ an orthonormal basis of a and notice that :

$$\langle a, b \rangle = \sum_{i=1}^l \langle aa_i, ba_i \rangle = \sum_{i=1}^l \langle a_i, ba_i \rangle = \sum_{i=1}^l \|ba_i\|^2,$$

the last sum being clearly at least 0 and at most l : it can take the value 0 iff $ba_i = 0$, for all $1 \leq i \leq l$, respectively the value l iff $ba_i = a_i$, for all $1 \leq i \leq l$, i.e. $a \subseteq b$ (also see Figure 1.10).

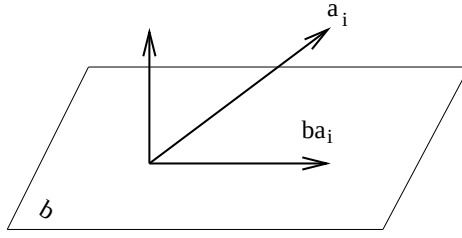


Figure 1.12: The decomposition of a_i with respect to the plane b

We will remark now that x_0 decomposes as $x_0 = \sum_{s=1}^N b_s$, where $b_s = \mathbb{R}^{s\perp}$, $1 \leq s \leq N$. Let us estimate now $h_{x_0}(a)$, for $a \in Z_\sigma$ arbitrary: For any $0 \leq k \leq n$ and any $s_k \leq s < s_{k+1}$, a can be expressed as $(a \cap \mathbb{R}^{s_k}) \oplus (a \cap \mathbb{R}^{s_k})^{\perp^a}$ (where $\perp^a$ denotes the orthogonal complement in a) in order to obtain: $\langle b_s, a \rangle = \langle b_s, a \cap \mathbb{R}^{s_k} \rangle + \langle b_s, (a \cap \mathbb{R}^{s_k})^{\perp^a} \rangle = \langle b_s, (a \cap \mathbb{R}^{s_k})^{\perp^a} \rangle \leq \dim(a \cap \mathbb{R}^{s_k})^{\perp^a} = n - \dim(a \cap \mathbb{R}^{s_k}) \leq n - k$ (s_0 and s_{n+1} must be readed as 1, respectively N). Consequently, $h_{x_0}(a) = \sum_{s=1}^N \langle b_s, a \rangle = \sum_{k=0}^n \sum_{s_k \leq s < s_{k+1}} \langle b_s, a \rangle \leq \sum_{k=0}^n (n - k)(s_{k+1} - s_k)$. Equality can occur iff $a \in C_\sigma$ and moreover, for any $0 \leq k \leq n$ and any $s_k \leq s < s_{k+1}$ holds $(a \cap \mathbb{R}^{s_k})^{\perp^a} \subseteq b_s$, hence for any $0 \leq k \leq n$, $(a \cap \mathbb{R}^{s_k})^{\perp^a} \subseteq (a \cap \mathbb{R}^{s_{k+1}-1})^\perp$, i.e. $a \cap \mathbb{R}^{s_{k+1}-1} \subseteq a \cap \mathbb{R}^{s_k}$, so, by reasons of dimension, $a \cap \mathbb{R}^{s_{k+1}-1} = a \cap \mathbb{R}^{s_k}$. The only n -plane $a \in C_\sigma$ which can satisfy all these conditions is $a = e_\sigma$.

□

The main result of the section is :

Theorem 1.7.2 *The map $f_\sigma : \hat{Z}_\sigma \rightarrow G_n(\mathbb{R}^N)$ is a cycle of Bott-Samelson type over $\mathbb{Z}_2$ for h_{x_0} at the critical point e_σ .*

PROOF It follows from the previous lemma and the fact that f_σ is invertible on C_σ that the function $h_{x_0} \circ f_\sigma : \hat{Z}_\sigma \rightarrow \mathbb{R}$ has a unique maximum at $f_\sigma^{-1}(e_\sigma) = \{\mathbb{R}e_{s_1}, \dots, \mathbb{R}e_{s_n}\} \stackrel{not.}{=} \{\phi_\sigma\}$. It remains to prove that:

- (i) the function $h_{x_0} \circ f_\sigma : \hat{Z}_\sigma \rightarrow \mathbb{R}$ has at ϕ_σ a nondegenerate critical point;
 - (ii) the index of e_σ as critical point of h_{x_0} equals $\dim \hat{Z}_\sigma$.
- (i) It will be enough to use the covering $\tilde{Z}_\sigma \rightarrow \hat{Z}_\sigma$ mentioned a few paragraphs above and to study the nondegeneracy at the point $(e_{s_1}, \dots, e_{s_n})$ of the function $\tilde{Z}_\sigma \rightarrow \mathbb{R}$ which associates to an arbitrary $(v_1, v_2, \dots, v_n) \in \tilde{Z}_\sigma$ the number:

$$\begin{aligned} h_{x_0} \circ f_\sigma(\mathbb{R}v_1, \dots, \mathbb{R}v_n) &= \sum_{s=1}^N \langle b_s, \text{Span}(v_1, \dots, v_n) \rangle = \\ &= \sum_{k=0}^n \sum_{s_k \leq s < s_{k+1}} \langle b_s, \text{Span}(v_{k+1}, \dots, v_n) \rangle = \\ &= \sum_{k=0}^n \sum_{s_k \leq s < s_{k+1}} \sum_{i=k+1}^n \langle b_s v_i, v_i \rangle = \\ &= \sum_{k=0}^n \sum_{s_k \leq s < s_{k+1}} \sum_{i=k+1}^n \|b_s v_i\|^2 = \\ &= \sum_{k=0}^n \sum_{s_k \leq s < s_{k+1}} \sum_{i=k+1}^n (1 - \|(b_s - I)v_i\|^2). \end{aligned}$$

Since the last expression is quadratic in the variables $v_1, v_2, \dots, v_n$, the nondegeneracy is quite clear.

(ii) We have already noticed in a more general context that, since the orbit $G_n(\mathbb{R}^N) = O(N).x_2$ is contained in a sphere of $S(N)$, the functions h_{x_0} and f_{-x_0} (i.e. the height-function with respect to x_0 , respectively the distance function from $-x_0$) differ only by an additive constant: it will be enough to calculate the index of e_σ regarded as critical point of f_{-x_0} . The last number can be computed by using Thm.1.6.2, being namely the number of focal hyperplanes crossed by the line segment $[-x_0, e_\sigma]$. This means that we only have to count how often two entries of different indices of the vector $-tx_0 + (1-t)e_\sigma$ can be equal, for $t \in (0, 1]$: a few simple calculations show that the number of times when such coincidences can occur equals $s_1 - 1 + s_2 - 2 + \dots + s_n - n$, and this is nothing but $\dim \hat{Z}_\sigma$.

□

We conclude by:

Corollary 1.7.3 *The Grassmann manifold $G_n(\mathbb{R}^N)$ is taut-embedded in the space $S(N)$ of symmetric $N \times N$ matrices. In their quality of cycles of Bott-Samelson type, the Schubert cycles $\{f_{\sigma*}[\hat{Z}_\sigma] : \sigma \text{ a Schubert symbol}\}$ constitute a basis of the space $H_*(G_n(\mathbb{R}^N))$.*

1.8 Schubert cycles on Grassmannians as projected HPT-cycles

This section is intended to give an explanation to the assertion made by Hsiang, Palais and Terng during their paper [19], which says that the cycles constructed by them there (and which we call HPT-cycles) are "generalized Schubert cycles". From our point of view, a relationship between these two objects can be obtained if we regard the Grassmann manifold as focal manifold asociated to a certain isoparametric submanifold and then compare the projected HPT-cycles considered in section 1.6 to the Schubert cycles considered in the previous section.

We have already explained how can be the Grassmannian considered as an isotropy orbit of the symmetric space $SL(N, \mathbb{R})/SO(N)$, as well as of $U(N)/O(N)$. Consequently, for any principal isotropy orbit of the latter one, all facts stipulated in Thm.1.2.3 will be supposed available. Before starting the direct approach, we notice that, by technical reasons, we will prefere to see this time $G_n(\mathbb{R}^N)$ not any more as the $O(N)$ -orbit of x_2 (see section1.7) but of $-x_2$. Since the following diagram is commutative,

$$\begin{array}{ccc}
 O(N).x_2 & \xrightarrow{h_{x_0}} & \mathbf{R} \\
 \downarrow \text{"_"} & & \downarrow \text{addition of a constant} \\
 O(N).(-x_2) & \xrightarrow{f_{x_0}} & \mathbf{R}
 \end{array}$$

the main result of our previous section can be transposed as follows: at any critical point e_σ of f_{x_0} corresponding to the Schubert symbol σ , the pair $(\hat{Z}_\sigma, f_\sigma)$ is a cycle of Bott-Samelson type for f_{x_0} at e_σ .

We will prefere that the role of the isoparametric submanifold which is going to be "focalized" to be played by the orbit of x_0 itself. This orbit is the *full flag manifold*, i.e. the classical flag manifold

$$F(N) = \{(L_1, \dots, L_N) : L_j \text{ line in } \mathbb{R}^N, L_i \perp L_j \text{ for any } 1 \leq i < j \leq N\}$$

. In the same manner as in the previous section, a flag of the form $(L_1, \dots, L_N)$ will be usually identified with the matrix of $S(N)$ which has the line L_j as

j -eigenspace, for all $1 \leq j \leq N$. In order to obtain concrete expressions for the HPT-cycles on $F(N)$ we need, of course, the concrete form of the curvature data (distributions, normals and, especially, spheres): these will be all obtained in the sequel by using Thm.1.2.3.

Because of the homogeneity of $F(N)$, it will be enough to restrict our attention to the point x_0 . The normal and the tangent spaces are:

$$\nu(F(N))_{x_0} = \mathfrak{a}, \quad T(F(N))_{x_0} = [x_0, \mathfrak{o}(N)].$$

The Weyl group is naturally isomorphic to the symmetric group $\text{cal}S_N$, which acts on the normal space $\mathfrak{a}$ by permuting the entries of the diagonal matrices. If $\alpha = \alpha_{ij}$, $j > i$, is an arbitrary positive root, then the value of the corresponding curvature normal at x_0 is

$$v_\alpha(x_0) = -\frac{1}{\alpha(x_0)}H_\alpha = \frac{1}{j-i}D_{ij},$$

where D_{ij} denotes the diagonal matrix whose entries are all 0, except those of index i , respectively j , which are 1, respectively -1 . At an arbitrary point $k.x_0 \in O(N).x_0$, we have $v_\alpha(k.x_0) = k.v_\alpha(x_0)$.

Denoting by E_{ij} the matrix whose entries are 0, except this situated on the line i and column j , which is 1, it turns out that the vector $y_\alpha = E_{ij} + E_{ji} \in TF(N)_{x_0}$ satisfies :

$$A_v(y_\alpha) = \langle v, v_\alpha \rangle y_\alpha,$$

for any $v \in \mathfrak{a}$. Because the curvature distributions are all 1-dimensional, it follows that :

$$E_\alpha(x_0) = \mathbb{R}(E_{ij} + E_{ji}) = \mathbb{R}[x_0, E_{ij} - E_{ji}].$$

Since $\mathfrak{k}_\alpha = \mathbb{R}(E_{ij} - E_{ji})$, it follows that K_α is the 1-parameter group

$$a_{ij}(t) = \exp(t(E_{ij} - E_{ji})) = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \cos t & \dots & \sin t & & \\ & & & \vdots & & \vdots & & \\ & & & -\sin t & \dots & \cos t & & \\ & & & & & & 1 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix},$$

with $t \in \mathbb{R}$. The curvature hyperplane $l_\alpha(x_0)$ is the same as $\text{Ker}\alpha$ and the corresponding curvature sphere $S_\alpha(x_0)$ is described by

$$t \rightarrow a_{ij}(t)x_0a_{ij}(t)^{-1},$$

$t \in \mathbb{R}$. At an arbitrary point $k.x_0 \in F(N)$, $l_\alpha(k.x_0) = k.l_\alpha(x_0)$ and $S_\alpha(k.x_0) = k.S_\alpha(x_0)$.

Geometrically, if $x = (L_1, \dots, L_N)$ is an arbitrary flag, then $a_{ij}.x$ is the flag obtained from x by a direct rotation of the orthogonal system of lines (L_i, L_j) in the 2-plane they generate. Even more general, if a is an arbitrary orthogonal transformation, we can interpret the flag $a.x$ as follows: if x is regarded as the symmetric matrix whose j -eigenspace is the line L_j , $1 \leq j \leq N$, then $a.x$ means the matrix axa^{-1} , i.e. the flag $(aL_1, \dots, aL_N)$. Similar geometric considerations can be made for partial flags (elements of $F(n) = O(N).x_1$, identified to n -flags of the form $(L_1, \dots, L_n)$ in $\mathbb{R}^N$) or for n -planes (elements of $G_n(\mathbb{R}^n) = O(N).x_2$).

Consider now τ an element of W together with the corresponding critical point $x = \tau x_0$ of the distance function $f_{x_0} : F(N) \rightarrow \mathbb{R}$. In order to construct the HPT-cycle at x we have first to relate the focal hyperplanes to x (since now they were related to x_0 , as the indices indicate). If α is an arbitrary root, then the hyperplane $\text{Ker}\alpha$ has the form $l'_\alpha(x)$ and the curvature sphere through x which focalizes at this hyperplane, call it $S'_\alpha(x)$, is given by:

$$t \mapsto a_{ij}(t)xa_{ij}(t)^{-1}, \quad t \in \mathbb{R}.$$

If we pass from x to an arbitrary $k.x$, $k \in O(N)$, the curvature sphere through $k.x$ which focalizes at $k.l'_\alpha(x)$ is $S'_\alpha(k.x)$, described by:

$$t \mapsto ka_{ij}(t)xa_{ij}(t)^{-1}k^{-1}, \quad t \in \mathbb{R}.$$

The line segment $[x_0, x]$ crosses exactly the focal hyperplanes $l'_{\alpha_{ij}}$ for which $i < j$ and $\frac{\tau(j) - \tau(i)}{j - i} < 0$. Going from x_0 to x , they are intersected in the order given by:

$$\frac{\tau(j_1) - \tau(i_1)}{j_1 - i_1} < \frac{\tau(j_2) - \tau(i_2)}{j_2 - i_2} < \dots < \frac{\tau(j_r) - \tau(i_r)}{j_r - i_r} < 0.$$

Denote $\alpha_{i_k j_k}$ simply by α_k and describe in this way the HPT-cycle of x as:

$$\mathcal{N}(x) = \{(f_1, \dots, f_r) : f_1 \in S'_{\alpha_1}(x), f_2 \in S'_{\alpha_2}(f_1), \dots, f_r \in S'_{\alpha_r}(f_{r-1})\},$$

together with $u : \mathcal{N}(x) \rightarrow F(N)$, $(f_1, \dots, f_r) \mapsto f_r$, for any $(f_1, \dots, f_r) \in \mathcal{N}(x)$. Its image in $F(N)$ is:

$$u(\mathcal{N}(x)) = \{a_{i_1 j_1}(s_1)a_{i_2 j_2}(s_2)\dots a_{i_r j_r}(s_r).x : s_1, s_2, \dots, s_r \in \mathbb{R}\}. \quad (1.11)$$

The idea is now to "project" the cycles of the form above from $F(N)$ to $F(n)$ and then to $G_n(\mathbb{R}^N)$. To this end we start with the vector $v_1(x_0) = x_1 - x_0 \in \nu(F(N))_{x_0}$, extend it to a parallel normal field v_1 on $F(N)$ and notice that the corresponding focal manifold is just $F(n)$. Denote by $\bar{\pi} : F(N) \rightarrow S(N)$ the associated end-point map. By Lemma 1.6.1, for any $\tau \in W$ holds $\bar{\pi}(\tau x_0) = \tau \bar{\pi}(x_0)$, or, equivalently, $\tau x_1 = \overline{\tau x_0}$, where we usually denote:

$$\bar{k} = \begin{cases} k, & \text{if } 1 \leq k \leq n \\ n+1, & \text{if } k \geq n+1, \end{cases}$$

whenever k is a positive integer. This time by Lemma 1.6.7, $\tau \in W$ is a minimal representative of W/W_{v_1} iff the line segments $[x_0, \tau x_0]$ and $[x_0, \tau x_1]$ intersect the same focal hyperplanes. Hence the set $\bar{S}_N$ of all these representatives can be explicitly described as follows:

$$\bar{S}_N = \{\tau \in S_N : \frac{\tau(i) - \tau(j)}{i - j} < 0 \text{ iff } \frac{\overline{\tau(i)} - \overline{\tau(j)}}{i - j} < 0, \text{ provided } 1 \leq i \neq j \leq N\}.$$

We can pass to $G_n(\mathbb{R}^N) = O(N).(-x_2)$ in a similar way: we consider $v_2(x_0) = x_0 + x_2$, extend it to a normal parallel field on $F(N)$ and regard $G_n(\mathbb{R}^N)$ as image of the end-point map $\pi_{v_2} \stackrel{not.}{=} \bar{\bar{\pi}} : F(N) \rightarrow S(N)$. If we associate this time to any positive integer k the number

$$\bar{\bar{k}} = \begin{cases} -1, & \text{if } 1 \leq k \leq n \\ 0, & \text{if } k \geq n+1, \end{cases}$$

then the set of minimal coset representatives of W/W_{v_2} is

$$\bar{\bar{S}}_N = \{\tau \in S_N : \frac{\tau(i) - \tau(j)}{i - j} < 0 \text{ iff } \frac{\overline{\overline{\tau(i)}} - \overline{\overline{\tau(j)}}}{i - j} < 0, \text{ provided } 1 \leq i \neq j \leq N\}.$$

The relation $\bar{\bar{S}}_N \subset \bar{S}_N$ which can be directly seen from the expressions above, is also clear by the fact that v_1 is "focalizes less" than v_2 , hence $W_{v_1} \subset W_{v_2}$.

Let us consider now an arbitrary critical point of $f_{x_0} : G_n(\mathbb{R}^N) \rightarrow \mathbb{R}$, of the form $x'' = \tau(-x_2)$, $\tau \in \overline{S}_N$. In order to apply Thm.1.6.9, we need the HPT-cycle $(\mathcal{N}(x), u)$ in $F(N)$ associated to $x = \tau x_0$, its image through $\bar{\pi}$ being a cycle of Bott-Samelson type at x'' for f_{x_0} . The focal hyperplanes intersected by $[x_0, \tau(-x_2))$, counted from x_0 to $\tau(-x_2)$, are these $l_{\alpha_{ij}}$ with $i < j$ for which $\frac{\overline{\tau(i)} - \overline{\tau(j)}}{i - j} < 0$, in the increasing order of the value of these expressions. Say that the roots get in this way are respectively $\alpha_{i_1 j_1}, \dots, \alpha_{i_r j_r}$, where $\tau(j_l) \in \{1, \dots, n\}$, $\tau(i_l) \in \{n+1, \dots, N\}$, for any $1 \leq l \leq n$ and $0 < j_1 - i_1 < j_2 - i_2 < \dots < j_r - i_r$. We partition the set $\{1, 2, \dots, N\}$ in two subsets, namely $J = \{\tau^{-1}(1), \dots, \tau^{-1}(n)\}$ and its complement $I = \{\tau^{-1}(n+1), \dots, \tau^{-1}(N)\}$. By ordering the elements of J one obtains something of the form

$$J = \{s_1 < s_2 < \dots < s_n\} \stackrel{not.}{=} \sigma,$$

which is a Schubert symbol. The decisive point of the section is the following lemma:

Lemma 1.8.1 *The image of the projected HPT-cycle $(\mathcal{N}(x), \bar{\pi} \circ u)$ in $G_n(\mathbb{R}^N)$ is contained in the image Z_σ of the Schubert cycle $(\hat{Z}_\sigma, f_\sigma)$.*

PROOF We will regard the element $\tau(-x_2)$ of $G_n(\mathbb{R}^N)$ as the n -plane $p_\tau = \bigoplus_{i=1}^n \mathbb{R}e_{s_i}$. Our aim is to prove that

$$a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_r j_r} \cdot p_\tau \in Z_\sigma,$$

where $a_{i_l j_l} = a_{i_l j_l}(t_l)$ with t_l arbitrary (see 1.11 above). Of course, it will be enough to prove that:

$$a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_r j_r} \cdot e_{s_i} \in \mathbb{R}^{s_i},$$

for any $1 \leq i \leq n$. An important role in proving that will play the following simple remark:

$$a_{ij} e_k = \begin{cases} e_k, & \text{if } k \neq i \text{ and } k \neq j \\ ce_i + de_j, & \text{if } k = i \text{ or } k = j, \end{cases} \quad (1.12)$$

where c and d are real numbers. We will prove now (1.12) by using the following algorithmic manner:

Step 1. We can evidently suppose $j_r = s_i$. It follows

$$a_{i_r j_r} e_{s_i} = c_1 e_{i_r} + d_1 e_{j_r},$$

where $i_r < j_r = s_i$.

Step 2. If $j_{r-1} > j_r$ then $i_{r-1} > i_r$, hence $a_{i_{r-1}j_{r-1}}$ doesn't change the expression from the previous step. So we can suppose $j_{r-1} \leq j_r$, consequently

$$a_{i_{r-1}j_{r-1}}a_{i_rj_r}.e_{s_i} = c'_1e_{i_r} + c'_2e_{i_{r-1}} + d'_1e_{j_r} + d'_2e_{j_{r-1}},$$

where $i_{r-1} < j_{r-1} \leq j_r = s_i$.

Step 3. If $j_{r-2} > j_r$ then $i_{r-2} > i_r$ and $i_{r-2} > i_{r-1}$, hence $a_{i_{r-2}j_{r-2}}$ doesn't change the expression from the previous step. So we can suppose $j_{r-2} \leq j_r$, consequently:

$$a_{i_{r-2}j_{r-2}}a_{i_{r-1}j_{r-1}}a_{i_rj_r}.e_{s_i} = c''_1e_{i_r} + c''_2e_{i_{r-1}} + c''_3e_{i_{r-2}} + d''_1e_{j_r} + d''_2e_{j_{r-1}} + d''_3e_{j_{r-2}},$$

where $i_{r-2} < j_{r-2} \leq j_r$.

.....
Final step. If $j_1 > j_r$, since $j_2, \dots, j_{r-1}$ is each at most j_r , it follows $i_1 > i_r$, $i_1 > i_{r-1}$, ..., $i_1 > i_2$, hence $a_{i_1j_1}$ doesn't change the expression from the previous step. So we can suppose $j_1 \leq j_r$, consequently

$$a_{i_1j_1}(a_{i_2j_2} \dots a_{i_rj_r}.e_{s_i}) = c_1^{(r)}e_{i_r} + \dots + c_r^{(r)}e_{i_1} + d_1^{(r)}e_{j_r} + \dots + d_r^{(r)}e_{j_1},$$

where $i_1 < j_1 \leq j_r$.

□

To the same $\tau \in \overline{\overline{S}}_N$ we associate now $x' = \tau x_1$ as critical point of the distance function on $F(n)$ from the base-point x_0 . By using a similar reasoning as before, we can prove that:

Lemma 1.8.2 *The image of the projected HPT-cycle $(\mathcal{N}(x), \bar{\pi} \circ u)$ in $F(n)$ is contained in $\hat{Z}_\sigma$.*

Remarks 1. Maybe it should be noticed that, formally, an another way to prove the two lemmas above would has been to begin with Lemma 1.8.2 and thereafter to use the relation $\overline{\overline{\pi}} = f_\sigma \circ \overline{\overline{\pi}}$ in order to deduce Lemma 1.8.1.

2. It seems difficult to get an appropriate understanding of the subsets $\overline{\overline{S}}_N$, respectively $\overline{\overline{S}}_N$ of S_N from a combinatorial point of view. Relevant in this sense could be for instance the bijective correspondance $\tau \mapsto \sigma = \{\tau^{-1}(1), \dots, \tau^{-1}(n)\}$ between $\overline{\overline{S}}_N$ and the set of Schubert symbols (it follows from here, for instance, that the cardinality of $\overline{\overline{S}}_N$ equals $\binom{N}{n}$). In our context,

this correspondance allows us to see precisely the identification of the critical point $\tau(-x_2)$ with the n -plane e_σ .

The main goal of the section is to show that the cycles $(\mathcal{N}(x), \overline{\pi} \circ u)$ and $(\hat{Z}_\sigma, f_\sigma)$ are the same in $H_*(G_n(\mathbb{R}^N))$. Before making the last step, we must notice an another interesting aspect:

Lemma 1.8.3 *The height function h_{-x_0} (and implicitly the distance function f_{x_0}) on $G_n(\mathbb{R}^N)$ has all points of the same index situated on the same level, for any possible value that the index could take.*

PROOF Consider $\tau \in \overline{S}_N$ arbitrarily, together with the corresponding critical point $x'' = \tau(-x_2) = e_\sigma = \text{Diag}(0, \dots, 0, -1, 0, \dots, 0, -1, 0, \dots, 0, -1, 0, \dots, 0)$, where the value -1 occurs successively in the positions $s_1, s_2, \dots, s_n$ dictated by σ . Its level is $h_{-x_0}(x'') = s_1 + \dots + s_n$. By Thm.1.7.2, its index equals $s_1 - 1 + s_2 - 2 + \dots + s_n - n = s_1 + s_2 + \dots + s_n - \frac{n(n+1)}{2}$ and the conclusion is clear. □

We are now in position to prove:

Proposition 1.8.4 *If $\sigma = \tau^{-1}\{1, 2, \dots, n\}$ and $x = \tau x_0$ are the Schubert symbol, respectively the critical point in $F(N)$ induced by an arbitrary $\tau \in \overline{S}_N$, then :*

$$(\overline{\pi} \circ u)_*[\mathcal{N}(x)] = f_{\sigma*}[\hat{Z}_\sigma].$$

PROOF We know already that both terms of the equality above are linking cycles for $h_{-x_0} : G_n(\mathbb{R}^N) \rightarrow \mathbb{R}$ at the critical point $x'' = \tau(-x_2)$ (cf. Thms. 1.6.9 and 1.7.2). We also know that all critical points of the same index as x'' are on the same level: that's why we can move x_0 inside $\Delta(x_0)$ to a vector y_0 in such a way that, with respect to h_{-y_0} , the level of x becomes strictly lower than the level of all other critical points of the same index (notice that, when passing from x_0 to y_0 inside $\Delta(x_0)$, nothing changes in the critical view of the associated distance function: the critical set is the same and also the pairs $(\mathcal{N}(x), \overline{\pi} \circ u)$, respectively $(\hat{Z}_\sigma, f_\sigma)$ are still cycles of Bott-Samelson type).

Let us express the Schubert cycle $f_{\sigma*}[\hat{Z}_\sigma]$ as sum of the projected HPT-cycles $(\overline{\pi} \circ u_i)_*[\mathcal{N}(\tau_i x_0)]$, $1 \leq i \leq r$, where $\tau_i(-x_2)$ are critical points of the same index as x'' , $\tau_i \in \overline{S}_N$. If $c = h_{-y_0}(x'')$, one knows that the space $H_*((G_n(\mathbb{R}^N)_c, \mathbb{Z}_2))$ is freely generated both by the Schubert cycles and the

projected HPT-cycles at the critical points of level at most c , in their quality of linking cycles (cf. Thm.1.1.4). On the other hand, the level of any $\tau_i(-x_2)$ which is different of x'' is strictly greater than c . This shows immediately the desired identity.

□

$$\begin{array}{ccc}
 \mathcal{N}(x) & \xrightarrow{\bar{\pi} \circ u} & \hat{Z}_\sigma \\
 & \searrow \bar{\pi} \circ u & \nearrow f_\sigma \\
 & Z_\sigma &
 \end{array}$$

From the commutativity of the diagram above and the previous proposition follows now:

$$(\bar{\pi} \circ u)_*[\mathcal{N}(x)] = f_{\sigma*}[\hat{Z}_\sigma] = f_{\sigma*}(\bar{\pi} \circ u)_*[\mathcal{N}(x)].$$

In particular, $(\bar{\pi} \circ u)_*[\mathcal{N}(x)] \neq 0$. But $\mathcal{N}(x)$ and $\hat{Z}_\sigma$ are both closed manifolds of the same dimension, hence $(\bar{\pi} \circ u)_*[\mathcal{N}(x)] = [\hat{Z}_\sigma]$. We conclude the section by:

Theorem 1.8.5 *To an arbitrary $\tau \in \bar{S}_N$ associate the Schubert cycle σ , the HPT-cycle $(\mathcal{N}(\tau x_0), u)$ in $F(N)$, the projected HPT-cycles $(\mathcal{N}(\tau x_0), \bar{\pi} \circ u)$ in $F(n)$, respectively $(\mathcal{N}(\tau x_0), \bar{\pi} \circ u)$ in $G_n(\mathbb{R}^N)$ and the Schubert cycle $f_\sigma : \hat{Z}_\sigma \rightarrow Z_\sigma$. Then the maps $\bar{\pi} \circ u : \mathcal{N}(\tau x_0) \rightarrow \hat{Z}_\sigma$ and $\bar{\pi} \circ u : \mathcal{N}(\tau x_0) \rightarrow Z_\sigma$ are both well-defined and surjective. The HPT-cycles $(\bar{\pi} \circ u)_*[\mathcal{N}(\tau x_0)]$ and $(\bar{\pi} \circ u)_*[\mathcal{N}(\tau x_0)]$ projected from $F(N)$ to $F(n)$, respectively $G_n(\mathbb{R}^N)$ coincide to $[\hat{Z}_\sigma]$, respectively the Schubert cycle $f_{\sigma*}[\hat{Z}_\sigma]$.*

Chapter 2

Cohomology of isoparametric submanifolds

The aim of this chapter is to continue the investigations initiated by Hsiang, Palais and Terng in the last section of their paper [19]. Their main idea was to adapt the old methods of Bott and Samelson (cf. [5]) to the theory they developed in the first sections of their paper. Concret results are obtained only for the cohomology ring over the field $\mathbb{Q}$, under the assumption of uniform multiplicity greater than 1: an essential ingredient in their reasoning was the using of Borel's formula concerning the rational cohomology of principal adjoint orbits (recall that the latter ones are isoparametric submanifolds of uniform multiplicity 2).

But maybe the most important idea of this section 6 of [19] is that the cohomological study of irreducible isoparametric submanifolds of uniform multiplicity greater than 1 can be reduced to the cohomological study of a prototype of this class, the best known being, of course, the adjoint orbit of a simple Lie group. It seems that the same "prototype principle" is available in a more general context, namely for the class of irreducible isoparametric submanifolds, divided in subclasses based on some natural criteria on the marked Dynkin diagrams. The first part of the chapter is devoted to proving this result. After "calculating" in this way the cohomology ring of isoparametric submanifolds, it will be very easy, by using the estimations made in section 1.6, to obtain the cohomology of an arbitrary focal manifold. Finally we will be interested about some quantitative estimations, mainly for the non-focal elements of isoparametric foliations: from each of the classes mentioned above, we will take an easy-to-handle exponent and try to say

as much as possible about its cohomology ring. Many such descriptions are taken from various sources: it was not easy to find the adequate information in the enormous literature about the topology of homogeneous spaces. Informations about the cohomology rings of all isoparametric hypersurfaces in spheres and their focal manifolds are given, for instance, in Münzner's paper [30]. By using our own methods, we will give exact descriptions of this rings.

The same convention as since now about the ring $\mathcal{R}$ of coefficients is still present during this chapter: $\mathcal{R} = \mathbb{Z}$, if both multiplicities m_1 and m_2 are greater than 1, otherwise $\mathcal{R} = \mathbb{Z}_2$ (also notice that in the first situation no torsion can occur in the spaces involved here, hence we can also find out everything about the homology and cohomology with $\mathbb{Z}_2$ -coefficients, simply by tensorizing with $\mathbb{Z}_2$). The convention is due to the considerations made in the first chapter about the orientability of HPT-cycles, so nothing more can be obtained by using only the classical theory of Morse.

2.1 The representation induced by W on $H_{lower}(M)$

We have already seen how the Weyl group associated to the isoparametric submanifold M acts on M . An important piece of information in the cohomological study of M is given by the representation that this action induces on the space:

$$H_{lower}(M) \stackrel{def.}{=} \begin{cases} H_{m_1}(M) \oplus H_{m_2}(M), & \text{if } m_1 \neq m_2 \\ H_m(M), & \text{if } m_1 = m_2 = m. \end{cases} \quad (2.1)$$

Before starting the direct approach, we would like to point out the following general fact, which concerns not only this section, but the entire chapter: it seems impossible to calculate the cohomology ring of an isoparametric submanifold if only regarded like an abstract object, as defined by Terng in [38] and studied also by her, together with Hsiang and Palais in [19]. The theorem of Thorbergsson is essential, because only using it we know that a certain symmetric space is involved and implicitly a root system, a well-defined Cartan matrix etc., all objects that will play an essential role, not only during our reasonings, but even in the final expressions. We will come back to this idea later, by supporting it with some concrete examples, but for the moment, it is worth to anticipate and say that the Weyl group

alone is in general not able to furnish enough information in order to calculate the ring $H^*(M)$, although it has been sufficient to say everything about the module $H_*(M)$.

That's why we begin with an irreducible symmetric space G/K of non-compact type, for which all considerations made during the second half of section 1.2 are, of course, available. Also now we consider $S = \{\gamma_1, \dots, \gamma_k\} \subset \Lambda$ a system of simple roots and take x_0 in the interior of the fundamental chamber $\Delta = \{h \in \mathfrak{a} : \gamma_i(h) \leq 0, \text{ for all } 1 \leq i \leq k\}$. The isotropy orbit $M = \text{Ad}(K).x_0$ is an irreducible isoparametric submanifold in $(\mathfrak{p}, B|_{\mathfrak{p}})$. We have already seen how the elements of Λ_0^+ can serve as indices for the curvature distributions, curvature spheres and curvature normals, under the assumption:

$$l_\lambda(x_0) = \text{Ker} \lambda,$$

for any $\lambda \in \Lambda_0^+$. The role of fundamental indices will be then played just by the simple roots.

Denote by b_i the homology class in M induced by the fundamental curvature sphere S_{γ_i} , $1 \leq i \leq k$. It is then quite clear that the module $H_{\text{lower}}(M)$ is freely generated by $b_1, \dots, b_k$. In the cases when we can take $\mathcal{R} = \mathbb{Z}$, the homology classes above are uniquely determined up to a sign, i.e. up to a changing of orientation of the spheres S_{γ_i} , $1 \leq i \leq k$. We associate to Λ the dual root system, consisting of the elements $\lambda^\vee = \frac{2\lambda}{\langle \lambda, \lambda \rangle}$, with $\lambda \in \Lambda$ and then, via S , the Cartan matrix of the latter one, that is the integers:

$$c_{ij} = \frac{2\langle \gamma_i^\vee, \gamma_j^\vee \rangle}{\langle \gamma_j^\vee, \gamma_j^\vee \rangle},$$

$1 \leq i, j \leq k$. We recall that they can also be defined by $s_{\gamma_i^\vee}(\gamma_j^\vee) = \gamma_j^\vee - c_{ji}\gamma_i^\vee$, for any pair of indices $1 \leq i, j \leq k$, hence they determine completely the representation of the Weyl group on the dual vector space $\mathfrak{a}^*$. Also consider the numbers d_{ij} , $1 \leq i, j \leq k$, which are given by:

$$d_{ij} = \begin{cases} 0, & \text{if } m_{\gamma_i} \neq m_{\gamma_j} \\ c_{ij} \bmod 2, & \text{if } m_{\gamma_i} = m_{\gamma_j} \text{ and } M \text{ has uniform multiplicity } 1, \\ c_{ij}, & \text{in all other cases,} \end{cases} \quad (2.2)$$

for $i \neq j$, and:

$$d_{ii} = \begin{cases} 0, & \text{if } m_{\gamma_i} \text{ is odd} \\ 2, & \text{if } m_{\gamma_i} \text{ is even.} \end{cases} \quad (2.3)$$

The representation induced by W on $H_{lower}(M)$ is completely determined if we know the matrix of the transformation induced by any generator φ_i , $1 \leq i \leq k$. The main result of the section is:

Theorem 2.1.1 *We can always choose an orientation of the fundamental curvature spheres $S_{\gamma_1}, \dots, S_{\gamma_k}$ such that:*

$$\varphi_{i*}(b_j) = b_j - d_{ji}b_i,$$

for all $1 \leq i, j \leq k$.

We prefer to treat separately the case when the rank of M is 2, mainly because of the case B_2 , when no root system can be, in general, involved, but also because we would like to consider here the non-irreducible type $A_1 \times A_1$. Of course, some superpositions with Thm.2.1.1 will occur in this way.

Theorem 2.1.2 *Let M be compact, complete, rank 2 isoparametric submanifold of multiplicities m_1 and m_2 and let the basis of $H_{lower}(M)$ induced by the principal curvature spheres S_1, S_2 be $\{b_1, b_2\}$, which is possibly determined only up to a sign. The representation of W on $H_{lower}(M)$ is given by:*

$$\varphi_{i*}(b_j) = b_j - d_{ji}b_i,$$

for any two $1 \leq i, j \leq 2$, where $d_{ii} \in \mathcal{R}$ are given by (2.3). Moreover, an orientation of the spheres S_1 and S_2 can be chosen, such that:

- a) If M is of type $A_1 \times A_1$, then $d_{12} = d_{21} = 0$.
- b) If M is of type A_2 , then $d_{12} = d_{21} = -1$.
- c) If M is of type B_2 , then, up to a converse,

$$(d_{12}, d_{21}) = \begin{cases} (-1, -2), & \text{if } m_1 = m_2 = 2 \\ (1, 0), & \text{if } m_1 = m_2 = 1 \\ (0, 0), & \text{in all other cases.} \end{cases}$$

- d) If M is of type G_2 , then, up to a converse,

$$(d_{12}, d_{21}) = \begin{cases} (-1, -3), & \text{if } m_1 = m_2 = 2 \\ (1, 1), & \text{if } m_1 = m_2 = 1. \end{cases}$$

Before starting to prove the theorems, we would like to notice some related facts:

Remarks 1. Since φ_i is the antipodal map in S_{γ_i} , it follows that $\varphi_{i*}(b_i)$ equals b_i or $-b_i$, depending on whether m_{γ_i} is odd, respectively, even.

2. The coefficients d_{ij} have to be readed in the corresponding $\mathcal{R}$. This remark regards for instance (2.3), where, even if m_{γ_i} is even, if it happens to have an another multiplicity equal to 1, d_{ii} must be 0, since readed in $\mathbb{Z}_2$.

3. One could say that the answer given for the B_2 case in Thm.2.1.2 is incomplete. In fact, ambiguities occur only when $(m_1, m_2) \in \{(1, 1), (2, 2)\}$. But in this latter situation, we have to do with an homogeneous M (cf. [8]), and in this case Thm.2.1.1 says everything. The same would be true about the G_2 case if we would dispose of a homogeneity result: but something like that is so far proved for uniform multiplicity 1 and only conjectured for uniform multiplicity 2 (cf. [11]). This things will affect in no way our future reasonings.

4. It is easy to notice which is the role that the marked Dynkin diagram of G/K can play in our developement: by Thm.2.1.1, in the homogeneous case it says everything about the action of W on $H_{lower}(M)$. At a closer analysis we can see that not all information given by the diagram is needed to this end. It will be enough to resume ourselves to what we call the *first coloured Dynkin diagram*; this one is obtained from the marked Dynkin diagram after colouring its nodes by black and white following the rules:

- a) if $m_1 = m_2 = 1$, figure that by I after the letter which indicates the case and colour all nodes by black;
- b) if $m_1 = m_2 \in \{2, 4, 8\}$, figure that by II and colour all nodes by white;
- c) if $m_1 \neq m_2$ and one of them equals 1, figure that case by III and colour the nodes of multiplicity m_1 by white and the others by black, or conversely;
- d) if $m_1 \neq m_2$ but both greater than 1, figure that case by IV and colour the nodes of even multiplicity by white and the other(s) by black;

then remove the exact values of the multiplicities; take into account that the $(BC)_k$ case is allways under the incidence of rule d), therefore superpose it to the $C_k IV$ case; also superpose $C_k III$ to $B_k III$. By using these rules, from the list of marked Dynkin diagrams we can immediately obtain the list of all possible first coloured Dynkin diagrams presented in Table 2 from Appendix.

5. We can restrict our attention exclusively to the coefficient ring $\mathbb{Z}_2$: if M is a principal isotropy orbit of the irreducible symmetric space G/K , then $H_{lower}(M, \mathbb{Z}_2)$ is always freely generated by the fundamental homology

classes $b_1, \dots, b_k$ of the fundamental spheres and the representation induced by W on $H_{lower}(M)$ is completely determined by:

$$\varphi_{i*}(b_j) = b_j + d'_{ji}b_i,$$

$1 \leq i, j \leq k$, where:

$$d'_{ji} = \begin{cases} 0, & \text{if } m_{\gamma_i} \neq m_{\gamma_j} \\ c_{ji} \bmod 2, & \text{otherwise.} \end{cases} \quad (2.4)$$

In order to recognize this representation, only the so-called *second coloured Dynkin diagram* of G/K is needed. It is obtained from the marked Dynkin diagram after colouring its nodes by black and white, as follows:

a') if $m_1 = m_2$, figure that by I after the letter which indicates the case and colour all nodes by white;

b') if $m_1 \neq m_2$, figure that by II and colour the nodes of multiplicity m_1 by white and the others by black, or conversely;

remove the values of the multiplicities; notice that $(BC)_k$ has only the chance to be followed by II, hence superpose $(BC)_k II$ to $B_k II$; it also makes no sense to differentiate between $B_k II$ and $C_k II$. The list of all possible second coloured Dynkin diagrams can be founded in Table 3 from Appendix.

6. As we have already anticipated, if M is an isoparametric submanifold, then the representation of W on $H_{lower}M$ cannot be, in general, readed, as long a certain symmetric space is not involved, together with the root system it carries. There exist isoparametric submanifolds which have the same Weyl group, but the representations of the latter one on the corresponding lower homology modules are different. In order to see that, but also in order to illustrate by concrete examples our method, let us analyse the following two situations, the Weyl group involved in both of them being of B_k type, k supposed at least 3.

(i) The marked Dynkin diagram is

$$B_k \quad \begin{array}{ccccccc} 1 & 1 & \dots & 1 & 1 & 1 \\ \circ & - & \circ & \dots & \circ & - & \circ & - & \circ & - & \circ \\ \gamma_1 & & \gamma_2 & & \gamma_{k-2} & & \gamma_{k-1} & & \gamma_k \end{array}$$

By Thm.2.1.1, the Cartan matrix modulo 2 of the dual root sytem (namely C_k) is needed, and this is:

$$(d_{ij}) = (c_{ij} \bmod 2) = \left(\frac{2\langle \gamma_i^\vee, \gamma_j^\vee \rangle}{\langle \gamma_j^\vee, \gamma_j^\vee \rangle} \bmod 2 \right) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

The general rule $\varphi_{i*}(b_j) = b_j + d_{ji}b_i$, $1 \leq i, j \leq k$ gives the representation of W on $H_1(M, \mathbb{Z}_2)$. Direct calculations made for $k = 3$ show that in this case the representation is faithful: concretely, it is sufficient to look at the matrices of w_* , $w \in W$, and to stop after obtaining 10 different such matrices, having in mind that $|W| = 18$. Probably the faithfulness of the representation remains valid for any value of k , but for the moment we are not able to prove that.

(ii) The marked Dynkin diagram is

$$C_k \quad \begin{array}{ccccccc} 1 & 1 & \dots & 1 & 1 & 1 \\ \circ & \circ & \dots & \circ & \circ & \circ \\ \gamma_1 & \gamma_2 & & \gamma_{k-2} & \gamma_{k-1} & \gamma_k \end{array}$$

The modulo 2 Cartan matrix of the dual root system is this time:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

and consequently, by the usual rule, φ_{k*} acts identically on $H_1(M, \mathbb{Z}_2)$. At least for $k = 3$, we have to do with a representation different from the one obtained at (i), although the Weyl group of both isoparametric submanifolds is the same, namely B_k .

7. The assumption made in section 1.5 about the orientation of the fundamental curvature spheres becomes only now clear. The only reason because these orientations have to be fixed is that we need a precise control of signs for the coefficients d_{ij} above.

It is now the moment to start the proof of the two theorems.

PROOF OF THEOREM 2.1.2 Without specializing yet to one of the types mentioned in the enunciation, take $i \in \{1, 2\}$ and denote by $\pi_i : M \rightarrow M_i$ the end-point map corresponding to the focal manifold M_i : more precisely, take $v_i(x_0) \in \nu(M)_{x_0}$ with $x_0 + v_i(x_0)$ belonging to $l_i(x_0)$ and no other focal hyperplane from $\nu(M)_{x_0}$ and put $\pi_i = \pi_{v_i}$. By Thm.1.2.2, π_i is a sphere bundle whose fiber through $x \in M$ is the m_i -sphere $S_i(x)$. If $\{j\} = \{1, 2\} \setminus \{i\}$, then, from the commutativity of the following diagram:

$$\begin{array}{ccc} & M & \\ & \pi_i \searrow & \\ \varphi_i & & M_i \\ & \pi_i \nearrow & \\ & M & \end{array}$$

it follows that $\pi_{i*}\varphi_{i*}(b_j) = \pi_{i*}(b_j)$, i.e. $\varphi_{i*}(b_j) - b_j \in \text{Ker}(\pi_{i*})$. As an immediate application of Thm.1.6.9, $\text{Ker}(\pi_{i*}|_{H_{\text{lower}}(M)}) = \mathcal{R}b_i$, so that there exists $d_{ji} \in \mathcal{R}$ with $\varphi_{i*}(b_j) = b_j - d_{ji}b_i$. A similar relation holds for $i = j$, with $d_{ii} \in \{0, 2\}$ (see Remark 2. from above), so it remains to determine d_{12} and d_{21} . It is the moment where we begin the case by case analysis.

a) M is of type $A_1 \times A_1$. The only curvature spheres are this time the fundamental ones, S_1 and S_2 . By Lemm1.5.1, $\varphi_2(S_1(x_0))$ has to be a curvature sphere through $\varphi_2(x_0)$. In order to decide which one is that, it is maybe enough to look at the picture 2.1. It follows $\varphi_2(S_1(x_0)) = S_1(\varphi_2(x_0))$.

b) M is of type A_2 . One knows in this case that $m_1 = m_2 \in \{1, 2, 4, 8\}$. From $(\varphi_1\varphi_2)^3 = id_M$ follows $\varphi_{1*}\varphi_{2*}\varphi_{1*}\varphi_{2*}(b_2) = \varphi_{2*}\varphi_{1*}(b_2)$. Since $\varphi_{i*}(b_i)$ is always $-b_i$, $i \in \{1, 2\}$, a few simple calculations lead to $d_{12}d_{21} = 1$, i.e. $d_{12} = d_{21} \in \{-1, 1\}$. The only question is now what happens when $m_1 = m_2 > 1$ and $d_{12} = d_{21} = -1$: in this case, from

$$\begin{cases} \varphi_{1*}(b_2) = b_2 - b_1 \\ \varphi_{2*}(b_1) = b_1 - b_2 \end{cases}$$

we are led to the right coefficients by simply changing the orientation of one of the fundamental spheres.

c) M is of type B_2 . We distinguish here the following three subcases:

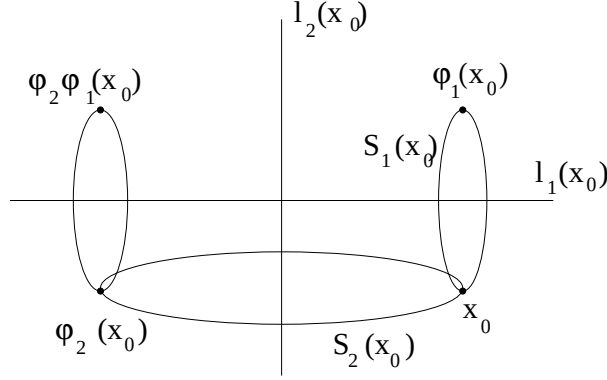


Figure 2.1: The case $A_1 \times A_1$.

c1) $m_1 \neq m_2$. There is nothing more to be proved.

c2) $m_1 = m_2 = 2$. By a result which goes back to E. Cartan, uniform multiplicity can be encountered in the B_2 case only if M is homogeneous. Uniform multiplicity 2 means M principal adjoint orbit of a compact connected Lie group of B_2 type (for instance $SO(3)$). That's why we must anticipate the proof of Thm.2.1.1, more exactly the 5th step of it, where the problem of adjoint orbits is solved. All it remains is to use correspondingly the Cartan matrix of the C_2 type, namely:

$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

c3) $m_1 = m_2 = 1$. Once again we use the fact that M is homogeneous, that is a principal isotropy orbit of the symmetric space $SO(5)/SO(2) \times SO(3)$. As we will see in a detailed manner at the 6th step of the proof of Thm.2.1.1, a such orbit can be identified to $S^1 \times SO(3) = S^1 \times \mathbf{P}\mathbb{R}^3$. From Lemma 1.5.1 and Figure 2.2 we can easily deduce that $\varphi_2\varphi_1S_2(x_0)$ and $\varphi_1S_2(x_0)$ represent the same curvature sphere.

Consequently $\varphi_{1*}(b_2) = \varphi_{2*}\varphi_{1*}(b_2)$, which is equivalent, after some elementary calculations, to $d_{12}d_{21} = 0$.

What we will show in the following, is that the alternative $d_{12} = d_{21} = 0$ can never occur. So, let us suppose that $\varphi_{1*}(b_2) = b_2$ and $\varphi_{2*}(b_1) = b_1$. We must now anticipate the method of calculating $H^*(M, \mathbb{Z}_2)$ which will be largely developed in section 2.3: the iterated sphere bundle F_4 considered there would have the cohomology ring

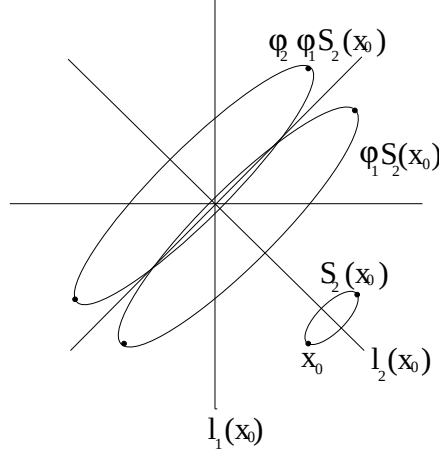


Figure 2.2: The case B_2 .

$H^*(F_4, \mathbb{Z}_2) = \mathbb{Z}_2[\xi_1, \xi_2, \xi_3, \xi_4]/(\xi_1^2, \xi_2^2, \xi_3^2, \xi_4^2)$. Since $H^1(M, \mathbb{Z}_2)$ is the subring of $H^1(F_4, \mathbb{Z}_2)$ generated by $\omega_1 = \xi_1 + \xi_3$ and $\omega_2 = \xi_2 + \xi_4$, it follows $\omega^2 = 0$, for any $\omega \in H^1(M)$. On the other hand, in $H^*(M) = H^*(\mathbf{P}\mathbb{R}^3) \times H^*(S^1)$ we can take the generator of the ring $H^*(\mathbf{P}\mathbb{R}^3, \mathbb{Z}_2)$, which is an element of order 4.

d) M is of type G_2 . There are two possibilities concerning the values of the multiplicities: $m_1 = m_2 = 2$ or $m_1 = m_2 = 1$. We treat them separately:

d1) $m_1 = m_2 = 2$. As long a proof of the homogeneity result in this case does not exist so far, we can not simply invoke Step 5 of the proof of Thm.2.1.1. We notice first that $\varphi_2\varphi_1 S_2(x_0)$ and $\varphi_1\varphi_2\varphi_1 S_2(x_0)$ represent curvature spheres with the same index (see Figure 2.3), consequently the homology classes $\varphi_{2*}\varphi_{1*}b_2$ and $\varphi_{1*}\varphi_{2*}\varphi_{1*}b_2$ are, up to a sign, the same. It turns out that, if this sign is "−", then $d_{12} = d_{21} \in \{-1, 1\}$ and then, in both cases, that $(\varphi_{1*}\varphi_{2*})^3$ is the identity on $H_2(M, \mathbb{Z})$. But that would contradict Lemma 6.9 of [19], which states just the faithfulness of the representation of W on $H_m(M, \mathbb{Z}_2)$ when $m_1 = m_2 = m > 1$. Consequently, the sign is "+" and it will be not difficult to deduce $d_{21}d_{12} = 3$.

d2) $m_1 = m_2 = 1$. This time it is quite clear that $\varphi_{2*}\varphi_{1*}b_2 = \varphi_{1*}\varphi_{2*}\varphi_{1*}b_2$ and this is equivalent to $d_{21}^2 d_{12} = d_{21}$. But similarly holds $d_{12}^2 d_{21} = d_{12}$. Hence we have to prove that d_{12} and d_{21} are not simultaneously 0. Suppose that this happens: by the same reasons as at c3), it follows $\omega^2 = 0$, for

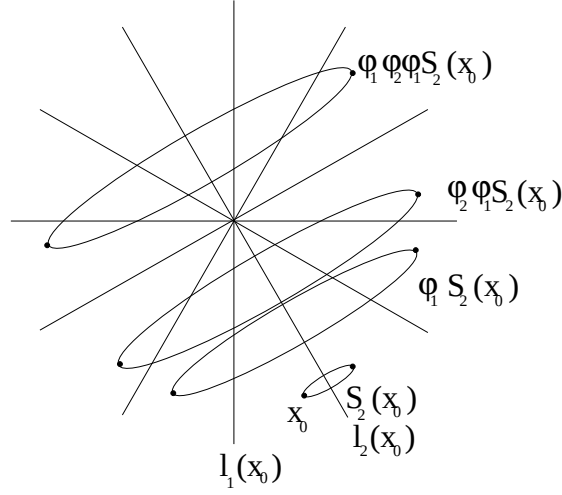


Figure 2.3: The case G_2 .

any $\omega \in H^1(M)$. In order to contradict this assertion, we will need some of the considerations made by Abresch in [1]: Denote by M' the image of M through the usual covering map $\Phi : S^7 \rightarrow \mathbf{P}\mathbb{R}^7$.

$$\begin{array}{ccc}
 M & \xrightarrow{\quad} & M' \\
 \downarrow & & \downarrow \\
 S^7 & \xrightarrow{\quad} & \mathbf{P}\mathbb{R}^7
 \end{array}$$

The restriction $\Phi|_M : M \rightarrow M'$ is also a two-fold covering (M being complete and totally geodesic) and the diagram above is commutative (the vertical arrows are the corresponding inclusions). If w is the image in $H^1(M')$ of the generator of $H^*(\mathbf{P}\mathbb{R}^7)$, then there exist $x_+, x_- \in H^2(M')$, such that

$$H^*(M') = \mathbb{Z}_2[x_+, x_-, w] / (w^4, x_+^2 + x_+ x_- + x_-^2, x_+^3, x_-^3) \quad (2.5)$$

Associate to the bundle $S^0 \rightarrow M \rightarrow M'$ its Gysin sequence and consider the following segment from it:

$$H^1(M') \xrightarrow{\cup w} H^2(M') \xrightarrow{\Phi^*} H^2(M).$$

Coming back to our supposition, from $\Phi^*(x_+)^2 = \Phi^*(x_+^2) = 0$ must follow now that $x_+^2 \in \text{Im}(\cup w)$. On the other hand, it is clear from (2.5) that something like that is not possible.

□

PROOF OF THEOREM 2.1.1 Something similar to what we said at the beginning of the previous proof is still available: for $i \in \{1, \dots, k\}$ arbitrarily chosen and fixed, take $v_i(x_0) \in \nu(M)_{x_0}$ such that $x_0 + v_i(x_0)$ belongs to the only focal hyperplane $l_i(x_0)$ from $\nu(M)_{x_0}$ and then $\pi_i = \pi_{v_i} : M \rightarrow \mathbb{R}^{n+k}$, whose image will be denoted by M_i . Once again we use the relation $\pi_i \circ \varphi_i = \pi_i$ in order to deduce that $\varphi_*(b_j) - b_j \in \text{Ker}(\pi_{i*}|_{H_{\text{lower}}(M)})$. Among the fundamental curvature spheres, only S_i has the property that the cycle it induces in $H_{\text{lower}}(M)$ is vanished through π_{i*} : we have $W_{v_i} = \{1, s_i\}$, hence for $j \neq i$, s_j is a minimal representative of W/W_{v_i} and we apply Thm.1.6.9. Consequently, for any two indices $i, j \in \{1, 2, \dots, k\}$, holds:

$$\varphi_{i*}(b_j) = b_j - d_{ji}b_i, \quad (2.6)$$

with $d_{ji} \in \mathcal{R}$.

We prefer from now on to divide the proof into several steps:

Step 1. If $\text{ord}(\varphi_i \varphi_j) = 2$, then d_{ji} given by (2.6) equals 0.

We have already noticed in section 1.5 (see Lemma 1.5.5) that if v is a parallel normal field with $x_0 + v(x_0)$ belonging to the only focal hyperplanes $l_i(x_0)$ and $l_j(x_0)$, then the connected component $S_{ij}(x_0)$ of x_0 in $\pi_v^{-1}(\pi_v(x_0))$ is a rank two isoparametric submanifold. The fundamental curvature spheres of the latter one, through any point z of it, are $S_i(z)$ and $S_j(z)$. It remains only to apply the point a) of Thm. 2.1.2. for $S_{ij}(x_0)$.

Step 2. If $\text{ord}(\varphi_i \varphi_j) = 3$, then d_{ji} given by (2.5) equals -1 , provided that the spheres S_i and S_j are oriented in an adequate way.

One uses the same reasoning as before.

Step 3. The cases A_k, D_k, E_k .

From the list of all possible marked Dynkin diagrams (see Table 1 from Appendix), it follows that these cases are always characterized by uniform multiplicity $m \in \{1, 2, 4, 8\}$. The previous two steps arrange almost everything; the only problem regards the eventual nonconcordance in choosing the orientations of $S_1, \dots, S_k$, provided $m > 1$. The standart form that a such accident could have is: $\varphi_{1*}(b_2) = b_2 + b_1$ and in the same time $\varphi_{3*}(-b_2) = -b_2 + b_3$.

But that also means $\varphi_{2*}(b_3) = b_3 - b_2$, so we only have to reverse the orientation of S_3 in order to obtain $\varphi_{3*}(b_2) = b_2 + (-b_3)$ and $\varphi_{2*}(-b_3) = -b_3 + b_2$. It is now easy to imagine a recursive method to choose orientations for fundamental curvature spheres, adequate in the sense of Thm.2.1.1.

Step 4. The case $(BC)_k$.

We only have to notice the form that the marked Dynkin diagram can have (particularly that m_1 is always different of m_2): apply Step 1., Step 2. and the reasoning above concerning a suitable choose of orientations for $S_1, \dots, S_{k-1}$.

Step 5. The action of W on $H_2(G/T, \mathbb{Z})$.

This is one of the main ingredients that Bott and Samelson used in [5] in order to calculate the the ring $H^*(G/T, \mathbb{Z})$. In the sequel we will reproduce their method, translated in the language of isoparametric submanifolds .

Let G be a compact, connected, simply connected and semisimple Lie group of Lie algebra $\mathfrak{g}$, $T \subseteq G$ a maximal torus and $\mathfrak{t}$ the Lie algebra of the latter one. Denote by $R \subset \mathfrak{t}^*$ the set of real roots of G with respect to T (as a basic reference can be consulted [6]). The Killing form B of $\mathfrak{g}$ is strictly negative definite , hence $(\mathfrak{g}, -B)$ is a Euclidean space. Also $\mathfrak{t}$ and consequently $\mathfrak{t}^*$ inherit an inner product and moreover, R becomes in a natural way a reduced root system in $\mathfrak{t}^*$. Choose $\{\gamma_1, \dots, \gamma_k\}$ a system of simple roots and associate R^+ the set of positive roots. For an arbitrary x_0 inside the Weyl chamber $\{h \in \mathfrak{t} : \gamma_i(h) > 0, \text{ for all } 1 \leq i \leq k\}$, consider the adjoint orbit $M = G.x_0$, which is the same as the homogeneous space G/T . We described in section 1.2, Remark 2., in which way can be M regarded as an isoparametric submanifold in $(\mathfrak{g}, -B)$. We would like to have now some more informations about that. If

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in R^+} \mathfrak{g}^\alpha$$

is the root space decomposition of $\mathfrak{g}$, then

$$[x_0, \mathfrak{g}] = \sum_{\alpha \in R^+} [x_0, \mathfrak{g}^\alpha]$$

is the decomposition of the tangent space of M at x_0 as sum of the curvature distributions. For any $\alpha \in R$, the space $\mathfrak{t} \oplus \mathfrak{g}^\alpha$ is the commutator of the hyperplane $\text{Ker} \alpha \subset \mathfrak{t}$, hence a subalgebra of $\mathfrak{g}$. The corresponding

Lie subgroup G_α of G is the $\text{Ad}G$ -centralizer of the same hyperplane. The (2-dimensional) curvature sphere through x_0 corresponding to the curvature distribution which has $E_\alpha(x_0) = [x_0, \mathfrak{g}^\alpha]$ as value at x_0 is clearly the orbit $G_\alpha.x_0 = G_\alpha/T$. The focal hyperplanes from $\nu(M)_{x_0} = \mathfrak{t}$ are $l_\alpha(x_0) = \text{Ker}\alpha$, $\alpha \in R^+$. When passing from x_0 to an arbitrary point $g.x_0$ of M , the curvature distributions, curvature spheres and focal hyperplanes are obtained by simply applying $\text{Ad}g$ respectively to the same objects corresponding to x_0 . The action of the Weyl group on M can be interpreted as given both by the antipodal maps as well as by:

$$k.(\text{Ad}g)(x_0) = (\text{Ad}g)(k.x_0),$$

for any $g \in G$ and any $k \in N_G(T)$.

The fundamental indices are $\gamma_1, \dots, \gamma_k$: if $\varphi_1, \dots, \varphi_k$ denote the corresponding elements of the Weyl group regarded as diffeomorphisms of M , our task is to express $\varphi_{i*}[G_{\gamma_j}/T]$, with $1 \leq i, j \leq k$, in terms of the basis $\{[G_{\gamma_l}/T] : 1 \leq l \leq k\}$ of $H_2(G/T, \mathbb{Z})$. In order to do that, we identify first $H_2(G/T, \mathbb{Z})$ to $\pi_2(G/T)$ by the Hurewicz isomorphism and then $\pi_2(G/T)$ to $\pi_1(T)$ by the boundary map ∂ of the homotopy sequence of the bundle $T \rightarrow G \rightarrow G/T$. The isomorphism $\phi : H_2(G/T, \mathbb{Z}) \rightarrow \pi_1(T)$ obtained in this way is W -invariant (cf. [5], Ch.IV, section 8).

Since the exponential $\exp|_{\mathfrak{t}} : \mathfrak{t} \rightarrow T$ is a covering map, we can identify $\pi_1(T)$ with the group of covering transformations, which can be further naturally identified to the integral lattice $I = \text{Ker}(\exp|_{\mathfrak{t}})$. But G was supposed simply connected, hence I is the same as the lattice generated by the inverse roots $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$, $\alpha \in R$ (cf. [6], Thm.7.1, Ch.V). Consequently, we can regard $\pi_1(T)$ as the free $\mathbb{Z}$ -module generated by the loops $\rho_i : s \mapsto \exp(s\gamma_i^\vee)$, $s \in [0, 1]$, $1 \leq i \leq k$.

On the other hand, for any $i \in \{1, \dots, k\}$ we can consider the bundle $T \rightarrow G_{\gamma_i} \rightarrow G_{\gamma_i}/T$ together with the following piece of the associated homotopy long exact sequence:

$$\pi_2(G_{\gamma_i}/T) \xrightarrow{\partial} \pi_1(T) \xrightarrow{j_\#} \pi_1(G_{\gamma_i}),$$

where j denotes the inclusion $T \hookrightarrow G_{\gamma_i}$. Since $\text{Ker}j_\#$ is the lattice generated by $\gamma_i^\vee$ (by the same [6], Thm.7.1, Ch.V), it follows that the map ϕ considered above associates to the fundamental homology class of the sphere G_{γ_i}/T the loop ρ_i , up to a sign. Choose the orientation of the fundamental curvature sphere G_{γ_i}/T in such a way, that $\phi[G_{\gamma_i}/T] = \rho_i$, for all $1 \leq i \leq k$.

Finally, we only have to notice that $\varphi_{i\sharp}(\rho_j)$ is the loop $s \mapsto \varphi_i \rho_j(s) = \varphi_i \exp(s\gamma_j^\vee) = \exp(\varphi_i(s\gamma_j^\vee)) = \exp(s(\gamma_j^\vee - c_{ji}\gamma_i^\vee))$, which is clearly homotopic to $\rho_j - c_{ji}\rho_i$.

Before passing to the next step, we would like to make a remark about the action of W on $H_2(G/T, \mathcal{R})$, when G is of B_k , respectively C_k type and $\mathcal{R}$ equals $\mathbb{Z}$ or $\mathbb{Q}$. From our previous Remark 6. follows that for $\mathcal{R} = \mathbb{Z}$ we have to do with two different representations of the same Weyl group W . As about $\mathcal{R} = \mathbb{Q}$, we can easily see that these representations are identic: For G of B_k type, from the associated marked Dynkin diagram

$$B_k \quad \begin{array}{ccccccc} 2 & 2 & & 2 & 2 & 2 \\ \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \circ \\ \gamma_1 & & \gamma_2 & & \gamma_{k-2} & \gamma_{k-1} & \gamma_k \end{array}$$

we can read:

$$\begin{cases} \varphi_{k*}(b_{k-1}) = b_{k-1} + b_k \\ \varphi_{k-1*}(b_k) = b_k + 2b_{k-1}. \end{cases} \quad (2.7)$$

If g is of C_k type, the sense of the arrow in the diagram above is reversed and consequently:

$$\begin{cases} \varphi_{k*}(b_{k-1}) = b_{k-1} + 2b_k \\ \varphi_{k-1*}(b_k) = b_k + b_{k-1}. \end{cases} \quad (2.8)$$

From our point of view, the only difference between the two situations consists only of these sets of relations and that can be easily annihilated by the following "change of variables":

$$b_i \mapsto 2b_i, \quad 1 \leq i \leq k-1, \quad \text{and} \quad b_k \mapsto b_k.$$

The coincidence of these two representations is not surprising if we take into account the classical W -equivariant isomorphism

$$H^*(G/T, \mathbb{Q}) \simeq \mathbb{Q}[x_1, \dots, x_k] / (\mathbb{Q}^+[x_1, \dots, x_k]^W),$$

where G is a compact connected rank k Lie group, $T \subseteq G$ a maximal torus, W the associated Weyl group and $\mathbb{Q}^+[x_1, \dots, x_k]^W$ the set of all homogeneous W -invariant polynomials of positive degree, $\deg(x_i) = 2$, $1 \leq i \leq k$ (cf. [3]).

Step 6. The case B_2 with $m_1 = m_2 = 1$

The action of W on $H_1(M, \mathbb{Z}_2)$ in this case is given by:

$$\begin{cases} \varphi_{1*}(b_2) = b_2 + b_1 \\ \varphi_{2*}(b_1) = b_1, \end{cases} \quad (2.9)$$

up to a converse (see Thm.2.1.2, especially its proof, point c3)). We intend in the sequel to give an exact answer.

The only noncompact symmetric space which can furnish our M is $SO_0(2, 3)/SO(2) \times SO(3)$. We recall that if $I_{2,3}$ denotes the diagonal matrix $\text{Diag}(-I_2, I_3)$, then the group $SO(2, 3) = \{x \in \mathbb{R}^{5 \times 5} : x^t I_{2,3} x = I_{2,3}, \det x = 1\}$ has two connected components and usually $SO_0(2, 3)$ denotes the component of the unit. The involutive automorphism $\sigma_{2,3} : x \mapsto I_{2,3} x I_{2,3}$, $x \in SO_0(2, 3)$, has $SO(2) \times SO(3)$ as fix point set. Without getting into details (they can be founded for instance in [18], Example I, §2, Ch.V), we only mention that, in the usual notations adopted in our section 1.2, $K = SO(2) \times SO(3)$ and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} : A \in \mathbb{R}^{2 \times 3} \right\},$$

K acting on $\mathfrak{p}$ by conjugation. As a maximal abelian subspace of $\mathfrak{p}$ can be chosen:

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & 0 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & 0 & \theta_2 \\ 0 & 0 & 0 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 & 0 \\ 0 & -\theta_2 & 0 & 0 & 0 \end{pmatrix} : \theta_1, \theta_2 \in \mathbb{R} \right\}.$$

The corresponding root system consists of $\pm\theta_1 \pm \theta_2, \pm\theta_1, \pm\theta_2$; as a simple root system we choose $\gamma_1 = \theta_1 - \theta_2, \gamma_2 = \theta_2$. Notice that $\|\gamma_1\| > \|\gamma_2\|$.

We fix now $x_0 \in \mathfrak{a}$ with $\gamma_1(x_0) > 0$ and $\gamma_2(x_0) > 0$ and associate the isoparametric submanifold $M = (SO(2) \times SO(3)).x_0$. The walls of the fundamental chamber are $l_{\gamma_1}(x_0) = \text{Ker}\gamma_1$ and $l_{\gamma_2}(x_0) = \text{Ker}\gamma_2$; as usual, the fundamental indices will be γ_1 and γ_2 . Expressed as a homogeneous manifold, M has the form

$$\frac{SO(2) \times SO(3)}{\Delta(\mathbb{Z}_2) \times \{1\}},$$

where $\Delta(\mathbb{Z}_2) \times \{1\}$ denotes the set $\{\text{Diag}(\varepsilon, \varepsilon, \varepsilon, \varepsilon, 1) : \varepsilon \in \{-1, 1\}\}$. We are also interested about the form that the focal manifold M_{γ_1} obtained after

focalization with the parallel normal field v_1 has , where $v_1(x_0) \in \nu(M)_{x_0}$ such that $x_0 + v_1(x_0)$ belongs to the only focal line $l_{\gamma_1}(x_0)$. A simple calculation shows:

$$M_{\gamma_1} = \frac{SO(2) \times SO(3)}{\Delta(SO(2)) \times \{1\}}$$

and moreover, the endpoint map $\pi_1 : M \rightarrow M_{\gamma_1}$ is just the natural projection induced by the inclusion $\Delta(\mathbb{Z}_2) \times \{1\} \hookrightarrow \Delta(SO(2)) \times \{1\}$. Decisive for the current step is the following claim:

Claim. The circle bundle $\pi_1 : M \rightarrow M_1$ is trivial. Consequently, $\varphi_{1*}(b_2) = b_2$ and $\varphi_{2*}(b_1) = b_1 + b_2$.

Before proving it, let us convince ourselves that the claim gives the right answer: The marked Dynkin diagram involved here is:

$$B_2 \quad \begin{array}{c} 1 \quad 1 \\ \circ \quad \circ \\ \gamma_1 \quad \gamma_2 \end{array}$$

hence the matrix we are expecting to use is

$$(d_{ij}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

in the sense that $\varphi_{i*}(b_j)$ must be $b_j + d_{ji}b_i$, $i, j \in \{1, 2\}$. We are evidently on the right way.

In order to prove the claim, we start with the following two diffeomorphisms:

$$\frac{SO(2) \times SO(3)}{\Delta(\mathbb{Z}_2) \times \{1\}} \simeq SO(2)/\mathbb{Z}_2 \times SO(3) = S^1 \times SO(3)$$

and

$$\frac{SO(2) \times SO(3)}{\Delta(SO(2)) \times \{1\}} \simeq SO(2)/SO(2) \times SO(3) = SO(3),$$

both given by the same correspondance law, namely:

$$[(g, k)] \mapsto ([g], k(g^{-1}, 1)),$$

for any $g \in SO(2)$, any $k \in SO(3)$ and equivalence classes adequate to each situation. They also furnish the following commutative diagram:

$$\begin{array}{ccc}
M & \xrightarrow{\pi_1} & M_{\gamma_1} \\
\downarrow & & \downarrow \\
S^1 \times SO(3) & \xrightarrow{\text{projection}} & SO(3)
\end{array}$$

which proves the first assertion. In order to prove the two relations, we can suppose that $M = M_{\gamma_1} \times S^1$ and $\varphi_1 : M_{\gamma_1} \times S^1 \rightarrow M_{\gamma_1} \times S^1$ maps an arbitrary pair $(x, z) \in M_{\gamma_1} \times S^1$ to $(x, -z)$. But then φ_1 is homotopic to the identity by $\varphi_t(x, z) = (x, e^{\pi i t} z)$, $(x, z) \in M \times S^1$, $t \in [0, 1]$, hence φ_{1*} equals the identity.

Step 7. The cases B_k and C_k with different multiplicities.

From the form that the Dynkin diagrams involved here can have, everything can be easily arranged by using only the first two steps.

Step 8. The cases B_k and C_k with $m_1 = m_2 = 2$.

We apply the previous Step 5.

Step 9. The cases B_k and C_k with $m_1 = m_2 = 1$.

We consider for instance the B_k case, namely we take G/K the symmetric space of noncompact type whose marked Dynkin diagram is that from below (without being interested about the exact expression of this space), together with the corresponding principal isotropy orbit $M = K.x_0$, $x_0 \in \mathfrak{a}$ regular.

$$B_k \quad \begin{array}{c} 1 \quad 1 \quad \cdots \quad 1 \quad 1 \quad 1 \\ \circ \text{---} \circ \cdots \circ \text{---} \circ \text{---} \circ \\ \gamma_1 \quad \gamma_2 \quad \gamma_{k-2}/k-1/k \end{array}$$

The first two steps say everything about the action of W on $H_{lower}(M)$, except the the values of $\varphi_{k*}(b_{k-1})$ and $\varphi_{k-1*}(b_k)$. In order to obtain them, as usual, we consider a corresponding rank 2 slice. More precisely, we take $v(x_0) \in \nu(M)_{x_0}$ such that $x_0^* \stackrel{not.}{=} x_0 + v(x_0) \in \Delta(x_0)$ is contained in the only focal hyperplanes $l_{\gamma_{k-1}}(x_0)$ and $l_{\gamma_k}(x_0)$, we extend it to a normal parallel field v and consider the slice $S_{x_0, v}$ through x_0 , i.e. the connectd component of $\pi_v^{-1}(\pi_v(x_0))$ which contains x_0 . In the same time, $S_{x_0, v}$ can be interpreted as connected component of the orbit $K_{x_0^*}.x_0$, given by the adjoint action of the stabilizer $K_{x_0^*}$ on $\mathfrak{p}_{x_0^*} = \{x \in \mathfrak{p} : [x, x_0^*] = 0\}$. It is clear that $S_{x_0, v}$ is an isoparametric submanifold in $\mathfrak{p}_{x_0^*}$. Maybe less clear is how could we get

a symmetric space whose principal isotropy orbit is $S_{x_0, v}$. An interesting geometric approach to this problem is given by Heintze and Olmos in [17]. We will not get into the details of their construction: relevant for us is only the fact that the isotropy representation of this symmetric space, call it P' , is the same as the adjoint representation of $(K_{x_0^*}/L_{x_0^*})_0$ on $\mathfrak{p}'$, where $L_{x_0^*} = \{k \in K_{x_0^*} : \text{Ad}(k) = \text{id on } \mathfrak{p}_{x_0^*}^*\}$ and $\mathfrak{p}' = \mathfrak{a}' + \sum_{\lambda(x_0^*)=0} \mathfrak{p}_\lambda$, the space $\mathfrak{a}'$ spanned by $H_{\gamma_{k-1}}$ and H_{γ_k} being maximal abelian in $\mathfrak{p}'$. In order to see our $S_{x_0, v}$ as an orbit of this representation, we must actually choose $v(x_0) \in \mathfrak{a}'$ (in fact, from the very beginning we can choose $-v(x_0)$ to be the orthogonal projection of x_0 onto $\mathfrak{a}' = (l_{\gamma_{k-1}} \cap l_{\gamma_k})_{\mathfrak{a}}^\perp$) and then consider the orbit of x_0 via the action of $(K_{x_0^*}/L_{x_0^*})_0$ on $x_0 + \mathfrak{p}'$. The roots of P' are the restrictions to $\mathfrak{a}'$ of the roots $\lambda \in \Lambda$ with $\lambda(x_0^*) = 0$. This means that P' is of B_2 type, i.e. $P' = SO_0(2, 3)/SO(2) \times SO(3)$ and $\gamma_{k-1}|_{\mathfrak{a}'}, \gamma_k|_{\mathfrak{a}'}$ form a system of simple roots. Since $\|\gamma_{k-1}\| > \|\gamma_k\|$, it follows from Step 6. that $\varphi_{k-1*}(b_k) = b_k$ and $\varphi_{k*}(b_{k-1}) = b_{k-1} + b_k$, Q.E.D.

Step 10. The case F_4 with $m_1 \neq m_2$.

We apply the results proved in the first two steps.

Step 11. The case F_4 with $m_1 = m_2 = 2$.

We have to do with an adjoint orbit, or this case is done by Step 5.

Step 12. The case F_4 with $m_1 = m_2 = 1$.

We apply Step 1, Step 2 followed by a reasoning similar to those made during Step 9.

Step 13. The case G_2 with $m_1 = m_2 = 2$.

We apply Step 5.

Step 14. The case G_2 with $m_1 = m_2 = 1$.

We are done by the point d) of Thm.2.1.2.

□

2.2 Homology and cohomology of F_p

We anticipate the results founded at the beginning of the next section (especially Lemma 2.3.1) by saying that a basic object in investigating the cohomology ring of M is the F-cycle associated to a minimal gallery of maximal length. We will come back temporarily to the general situation considered in our first chapter: M^n will be in this section an arbitrary irreducible, compact

and full isoparametric submanifold of $\mathbb{R}^{n+k}$, without any assumption of homogeneity. As usual, we fix $x_0 \in M$ together with its Weyl chamber $\Delta(x_0)$ (distinguished as the fundamental one) and its walls $l_1(x_0), \dots, l_k(x_0)$. Let w_0 be the longest element of the Weyl group W . To an arbitrary reduced expression of w_0 we associate the corresponding minimal gallery, say it of type $f = i_1 i_2 \dots i_p$. This gallery ends clearly at the chamber corresponding to $w_0 = s_f$, namely the opposite of $\Delta(x_0)$ and its length p equals the number of focal hyperplanes. Denote the corresponding F-cycle by (F_p, u) , with

$$F_p = F_f = \{(z_1, \dots, z_p) \in M^{\times p} : z_1 \in S_{i_1}(x_0), z_2 \in S_{i_2}(z_1), \dots, z_p \in S_{i_p}(z_{p-1})\}$$

and $u = u_f : F_p \rightarrow M$, $(z_1, \dots, z_p) \mapsto z_p$, $(z_1, \dots, z_p) \in F_p$. Since $w_0 x_0$ is the maximum point of the distance function f_{x_0} on M , it follows that the cycle $u_{f*}[F_f]$ must be the generator of the top-homology module of M , in its quality of cycle of linking type at $w_0 x_0$. In particular, F_p is a closed manifold of the same dimension as M and u is a map of degree ± 1 between F_p and M .

The study of F_p is facilitated by the fact that it is the total space of an iterated bundle, whose first base space and fibers at each level are spheres, each bundle involved here admitting a section. More precisely, for any $1 \leq r \leq p$, take

$$F_r = \{(z_1, z_2, \dots, z_r) \in M^{\times r} : z_1 \in S_{i_1}(x_0), z_2 \in S_{i_2}(z_1), \dots, z_r \in S_{i_r}(z_{r-1})\}.$$

Then $\pi_r : F_r \rightarrow F_{r-1}$, $(z_1, \dots, z_r) \mapsto (z_1, \dots, z_{r-1})$, $(z_1, \dots, z_r) \in F_r$, is an $S^{m_{i_r}}$ -bundle and $s_r : F_{r-1} \rightarrow F_r$, $(z_1, \dots, z_{r-1}) \mapsto (z_1, \dots, z_{r-1}, z_{r-1})$, $(z_1, \dots, z_{r-1}) \in F_{r-1}$ is a section of it. The aim of this section is to use all these particularities in order to prove that, from the homological and, up to a certain point, cohomological viewpoints, F_p has the same behaviour as the product of spheres $S^{m_{i_1}} \times \dots \times S^{m_{i_p}}$.

We fix for the moment $r \in \{1, 2, \dots, p\}$ and denote by $\mathcal{I}_r$ the set $\{1, 2, \dots, r\}$. To any $I \subseteq \mathcal{I}_r$ of the form $I = \{j_1 < \dots < j_t\}$, we associate the subspace

$$a_I^r = \{(z_1, \dots, z_r) \in M^{\times r} : z_1 = \dots = z_{j_1-1} = x_0, z_{j_1} = \dots = z_{j_2-1} \in S_{i_{j_1}}(x_0), \\ z_{j_2} = \dots = z_{j_3-1} \in S_{i_{j_2}}(z_{j_1}), \dots, z_{j_t-1} = \dots = z_{j_t-1} \in S_{i_{j_{t-1}}}(z_{j_{t-2}}), z_{j_t} = \dots = z_r \in S_{i_{j_t}}(z_{j_{t-1}})\}$$

of F_r . By convention, a_ϕ^r means the set consisting of one point. By an inductive argument, we will prove first:

Proposition 2.2.1 *The set $\{a_I^r : I \subseteq \mathcal{I}_r\}$ form a basis of the module $H_*(F_r)$.*

The assertion is quite clear for $r = 1$. In order to make the induction step, we consider the bundle $\pi_r : F_r \rightarrow F_{r-1}$, together with the section $s_r : F_{r-1} \rightarrow F_r$ described above and use the following two lemmas:

Lemma 2.2.2 *The map s_{r*} is injective and $H_*(F_r) = \text{Im}s_{r*} \oplus \text{Ker}\pi_{r*}$.*

PROOF We are resumed to a purely algebraic fact: $A = H_*(F_r)$ and $B = H_*(F_{r-1})$ are Abelian groups, $\pi = \pi_{r*} : A \rightarrow B$ and $s = s_{r*} : B \rightarrow A$ are morphisms of groups with $\pi \circ s = \text{id}_B$. It must follow $A = \text{Im}(s) \oplus \text{Ker}(\pi)$. We notice that if $p : A \rightarrow \text{Ker}\pi$, $a \mapsto a - s(\pi(a))$, $a \in A$ and $i : \text{Ker}\pi \hookrightarrow A$ is the inclusion map then $p \circ i = \text{id}_{\text{Ker}\pi}$, $p \circ s = 0$, $\pi \circ i = 0$ and $i \circ p + s \circ \pi = \text{id}_A$. Consequently, we can use [10], Prop.2.16, Ch.I, and obtain the desired equality. □

Lemma 2.2.3 *For any $I \subseteq \mathcal{I}_r$ holds:*

- a) *If $r \notin I$, then $\pi_{r*}(a_I^r) = a_I^{r-1}$ and $s_{r*}(a_I^{r-1}) = a_I^r$.*
- b) *If $r \in I$, then $\pi_{r*}(a_I^r) = 0$.*

$$\begin{array}{ccc} a_I^r & \xrightarrow{\quad} & a_{I \setminus \{r\}}^{r-1} \\ & \searrow \pi_r \quad \swarrow & \\ & F_{r-1} & \end{array}$$

PROOF The only fact that really needs a proof is the assertion made at point b). Consider the commutative diagram from below, where the arrow which starts at $a_{I \setminus \{r\}}^{r-1}$ represents the inclusion map and the horizontal arrow is described by the same law as π_r . Since $\dim a_{I \setminus \{r\}}^{r-1} < \dim a_I^r$, the conclusion follows immediately. □

In order to accomplish the induction step, we must consider the Gysin homology sequence of the bundle $S^{m_{i_r}} \rightarrow F_r \xrightarrow{\pi_r} F_{r-1}$, namely:

$$\dots \rightarrow H_{*+m_{i_r}+1}(F_{r-1}) \rightarrow H_*(F_{r-1}) \xrightarrow{\tau} H_{*+m_{i_r}}(F_r) \xrightarrow{\pi_{r*}} H_{*+m_{i_r}}(F_{r-1}) \rightarrow \dots$$

in order to deduce that $\text{Ker}\pi_{r*} = \text{Im}\tau$. Whenever I is a subset of $\mathcal{I}_r$ with $r \in I$, we take the Gysin sequence of the bundle $S^{m_{i_r}} \rightarrow a_I^r \xrightarrow{\pi_r} a_{I \setminus \{r\}}^{r-1}$, namely the following piece:

$$H_{\dim(a_{I \setminus \{r\}}^{r-1})}(a_{I \setminus \{r\}}^{r-1}) \xrightarrow{\tau} H_{\dim(a_I^r)}(a_I^r) \xrightarrow{\pi_{r*}} H_{\dim(a_I^r)}(a_{I \setminus \{r\}}^{r-1}) = 0$$

and deduce that $\tau[a_{I \setminus \{r\}}^{r-1}] = \pm[a_I^r]$. Finally we invoke the naturality of the transfer map τ .

Remark 1. Remember that, by our convention, $\mathcal{R} = \mathbb{Z}$ iff the values of both multiplicities are greater than 1. In this situation, all manifolds involved above (namely F_r and a_I^r , with $1 \leq r \leq p$ and $I \subseteq \mathcal{I}_r$) are simply connected (as can be seen inductively by using the homotopy long exact sequence), so orientable: their precise orientation is given by the assumption made in section 1.5 and became concrete later, in section 2.1, Remark 7.

We are interested now about the cohomology ring of F_p . At any iteration $\pi_r : F_r \rightarrow F_{r-1}$ we consider $\xi_r \in H^{m_{i_r}}(F_r)$ uniquely determined by: $\xi_r(a_r^r) = 1$ and $\xi_r(a_j^r) = 0$, for any $1 \leq j < r$ with $m_{i_j} = m_{i_r}$. Of course, the restriction of ξ_r to any fiber generates, together with the unit 1, the cohomology of this fiber. It follows from the theorem of Leray-Hirsch that the map $\pi_r^* : H^*(F_{r-1}) \rightarrow H^*(F_r)$ is injective and

$$H^*(F_r) = \pi_r^* H^*(F_{r-1}) \oplus \xi_r \pi_r^* H^*(F_{r-1}). \quad (2.10)$$

Via the monomorphisms of the form π_{j*} , we will interpret all rings $H^*(F_k)$, $1 \leq k < r$ as subrings of $H^*(F_r)$: in particular, any ξ_j with $1 \leq j \leq r$ is regarded as an element of $H^*(F_r)$.

Definition Let A^* be a sqew-commutative $\mathcal{R}$ -algebra (i.e. $xy = (-1)^{\deg(x)\deg(y)}yx$, for any two $x, y \in A^*$) We say that A^* is *simply generated* by its elements $x_1, x_2, \dots, x_r$ and denote that by $A^* = \Delta(x_1, \dots, x_r)$ if:

- a) as $\mathcal{R}$ -algebra, A^* is generated by $x_1, x_2, \dots, x_r$;
- b) as $\mathcal{R}$ -module, A^* has a basis consisting of the products $x_J \stackrel{\text{not.}}{=} x_{j_1} x_{j_2} \dots x_{j_t}$, with $J = \{j_1 < j_2 < \dots < j_t\} \subseteq \mathcal{I}_r$.

An algebra A^* simply generated by $x_1, x_2, \dots, x_r$ is clearly completely determined only when all squares x_j^2 , $1 \leq j \leq r$, can be expressed as linear combinations of products of the form $x_{j_1} x_{j_2}$, with $j_1 < j_2$. As a particular

case, if $A^* = \Delta(x_1, x_2, \dots, x_r)$ with $x_j^2 = 0$, for any $1 \leq j \leq r$, then A^* is the exterior algebra $\Lambda(x_1, \dots, x_r)$.

It follows from the previous considerations that $H^*(F_r) = \Delta(\xi_1, \dots, \xi_r)$, with $\deg \xi_j = m_{i_j}$, $1 \leq j \leq r$. So the only missing piece of information is the expression of ξ_j^2 as linear combination of monomials of the form $\xi_{j_1} \xi_{j_2}$, $1 \leq j_1 < j_2 \leq r$. We will solve this problem in our next section. For the moment, we are interested about another aspect, namely the duality between the basis $\{\xi_J = \xi_{j_1} \dots \xi_{j_t} : J = \{j_1 < \dots < j_t\} \subseteq \mathcal{I}_r\}$ of $H^*(F_r)$ and the basis $\{a_J^r : J \subseteq \mathcal{I}_r\}$ of $H_*(F_r)$. We will solve this problem in two steps as follows:

Proposition 2.2.4 *For any subset J of $\mathcal{I}_r$, holds $\xi_J(a_J^r) = (-1)^{l(J)}$, where:*

$$l(J) = \begin{cases} 0, & \text{if } J \text{ consists of only one element} \\ \sum \{m_{i_{j_1}} m_{i_{j_2}} : j_1, j_2 \in J, j_1 < j_2\}, & \text{otherwise.} \end{cases}$$

PROOF It will be sufficient to prove that $\xi_{\mathcal{I}_r}([F_r]) = (-1)^{l_r}$, where l_r denotes 0 if $r = 1$, respectively $\sum_{1 \leq j_1 < j_2 \leq r} m_{i_{j_1}} m_{i_{j_2}}$, otherwise. We will do that by an inductive argument. For $r = 1$, the equality is evident.

We show now that if ξ denotes simply $\xi_{\mathcal{I}_{r-1}}$, then from $\xi[F_{r-1}] = (-1)^{l_{r-1}}$ follows $\xi_{\mathcal{I}_r}([F_r]) = \pi_r^*(\xi) \xi_r([F_r]) = (-1)^{l_r}$. As usual, $a_r^r \in H_{m_{i_r}}(F_r)$ denotes the fundamental homology class of the fiber in the bundle $\pi_r : F_r \rightarrow F_{r-1}$. Let ξ' denote the Poincaré dual of a_r^r in $H^*(F_r)$. Since the submanifold $s_r(F_{r-1})$ and a fiber intersect at only one point, the intersection number being +1 (see Remark 1, section 1.5), it follows from [14], Prop.31.7, that

$$\xi'(s_{r*}[F_{r-1}]) = (-1)^{q_r}, \quad (2.11)$$

where $q_r = \dim a_r^r \cdot \dim F_{r-1} = m_{i_r} (\sum_{j=1}^{r-1} m_{i_j})$.

On the other hand, one can easily see that $\pi_{r*}([F_r] \cap \pi_r^*(\xi)) = 0$, hence by Lemma 2.2.3, $[F_r] \cap \pi_r^*(\xi) = k a_r^r$, with $k \in \mathbb{Z}$. But then $k = \xi_r([F_r] \cap \pi_r^*(\xi)) = (\pi_r^*(\xi) \cup \xi_r)([F_r]) = (-1)^l$, for a certain integer l (we used in the last step the theorem of Leray-Hirsch). Consequently, $\pi_r^*(\xi) = (-1)^l \xi'$ and by (2.11), $l = q_r + l_{r-1}$, i.e. $l = l_r$.

□

The second step is made by:

Proposition 2.2.5 *Whenever I and J are different subsets of $\mathcal{I}_r$ with $\dim a_I^r = \dim a_J^r$, holds $\xi_J(a_I^r) = 0$.*

PROOF Once again we will use an inductive argument by $r \geq 1$. For $r = 1$, everything is clear. Let us suppose that the assertion is true for any $1 \leq k < r$ and prove it for r .

Case 1: $r \notin J$. If $r \notin I$, then $\xi_J(a_I^r) = \pi_r^*(\xi_J)(s_{r*}(a_I^{r-1})) = \xi_J(\pi_{r*}s_{r*}(a_I^{r-1})) = \xi_J(a_I^{r-1})$ and we can use the induction hypothesis. If $r \in I$, then $\xi_J(a_I^r) = \pi_r^*(\xi_J)(a_I^r) = \xi_J(\pi_{r*}a_I^r) = 0$, by Lemma 2.2.3.

Case 2: $r \in J$. Take J of the form $J = J' \cup \{r\}$, with $J' \subset \mathcal{I}_{r-1}$.

We have to prove that $\xi_J(a_I^r) = \xi_r \cup \xi_{J'}(a_I^r) = 0$.

If $J' \not\subseteq I$, notice that $\xi_J(a_I^r) = (\xi_r \cup \xi_{J'})(i_{I*}^r[a_I^r]) = (i_I^r)^*(\xi_r) \cup (i_I^r)^*(\xi_{J'})([a_I^r])$, where $i_I^r : a_I^r \hookrightarrow F_r$ is the inclusion map. The idea is to show that the element $(i_I^r)^*(\xi_{J'})$ of $H^*(a_I^r)$ is zero. As long as the module $H_*(a_I^r)$ is freely generated by the cycles a_K^r , with $K \subseteq I$, we only need to take a such K with $\dim a_K^r = \dim a_{J'}^r$ and to observe that $(i_I^r)^*(\xi_{J'})(a_K^r) = \xi_{J'}(a_K^r)$ is zero, by using Case 1. (it is impossible to have $K = J'$).

If $J' \subseteq I$ and $r \notin I$, then $(\xi_r \cup \xi_{J'})(a_I^r) = (\xi_r \cup \xi_{J'})(s_{r*}a_I^{r-1}) = s_r^*(\xi_r) \cup s_r^*(\xi_{J'})(a_I^{r-1}) = 0$, since $s_r^*(\xi_r) = 0$ (ξ_r has been chosen in such a way that it vanishes any m_{i_r} -dimensional element of $\text{Im}s_{r*}$).

From $J' \subseteq I$ and $r \in I$ would follow $J \subsetneq I$, hence $\dim a_J^r < \dim a_I^r$, impossible.

□

2.3 The ring $H^*(M_v, \mathcal{R})$

If our object of study, the manifold M , was considered in the previous section from the abstract perspective of isoparametric submanifolds, we must now come back to the position adopted in the first section of the present chapter: M will be again a principal isotropy orbit of a certain irreducible symmetric space or a rank 2 irreducible isoparametric submanifold (this two viewpoints of seeing M are, of course, identic). Our goals in this section are:

- a) to elucidate completely the structure of the cohomology ring of F_p ;
- b) to relate the cohomology rings of M and F_p , sufficiently close in order to be able to use point a) and give a precise description of $H^*(M)$;
- c) to obtain the cohomology ring of an arbitrary focal manifold M_v .

As we already declared at the beginning of the chapter, the start-point of our investigations is the paper [19] of Hsiang, Palais and Terng. In an incipient stage, many ideas of our present developement can be founded in

the 6-th section of this paper. Maybe some concrete delimitations have to be made:

a') In order to calculate the cohomology of the highest dimensional F-cycle, we will use the same method as the authors above used to calculate the cohomology of the highest dimensional HPT-cycle (actually, we have to look for the origins of the idea in Bott and Samelson's paper [5]). Their convention concerning the choice of the ring $\mathcal{R}$ is just slightly different of ours, in the sense that they allowed $\mathcal{R}$ to be $\mathbb{Z}$ only if the two multiplicities are equal and greater than one, and that, only in order to gain the commutativity of the rings $H^*(M)$ and $H^*(F_p)$. By our method, we will manage also with the integral cohomology in the case $(BC)_k IV = C_k IV$ (see Table 2. from Appendix). On each level F_r of the iterated bundle acts naturally a certain element of the Weyl group: the action induced on $H_{lower}(F_r)$ plays here an important role and can be directly deduced from what we already know about the action of W on $H_{lower}(M)$. That's not the case in the paper [19]: very similar descriptions of these two representations are given also there, but they are not complete (now we know that it would have been impossible to get these descriptions completely, as long as a certain root system was not involved).

b') In relating M and the highest dimensional HPT-cycle, firstly from the homological viewpoint, Hsiang, Palais and Terng restricted themselves only to H_{lower} . By dualizing this relationship, they could, of course, only relate the lower cohomology modules. That's a too small piece of information to furnish general descriptions: only under the severe restrictions $m_1 = m_2 \geq 2$ and $\mathcal{R} = \mathbb{Q}$ concrete estimations have been achieved (and that is because their reasoning is essentially based on the result of Borel result which says that the rational cohomology ring of a principal adjoint orbit is generated by its degree two elements). At our turn, we will be able to obtain a closer relationship between the homology modules of M and our highest dimensional F-cycle, thanks to the descriptions we gave to these two modules in our previous sections 1.5, respectively 2.2. By using the duality stated in Prop.2.2.1, this relationship will be immediately transposed in cohomology.

c') The cohomology ring of a focal manifold is obtained in [19] under the same severe restrictions described above (uniform multiplicity greater than 1 and rational cohomology): the method is still based on Borel's results and a dimension-comparing argument. In a totally different way, we will be led to general estimations. Namely, we will transpose in cohomology the results of section 1.6, especially the surjectivity of π_{v*} , which implies the injectivity

of π_v^* and describe the image of π_v^* in $H^*(F_p)$, the latter one being already known, by a).

Let us begin now with a simple topological consequence of the fact already noticed at the beginning of section 2.2, which says that the map $u : F_p \rightarrow M$ is of degree ± 1 :

Lemma 2.3.1 *The map $u^* : H^*(M) \rightarrow H^*(F_p)$ is injective and its image is a direct summand of the free $\mathcal{R}$ -module $H^*(F_p)$.*

PROOF (cf. [5]). We denote both Poincarè duality maps associated to M and F_p by d (they are actually isomorphisms). Since u has degree ± 1 , the following diagram is, up to a sign, commutative:

$$\begin{array}{ccc} H^*(M) & \xrightarrow{u^*} & H^*(F_p) \\ \downarrow d & & \downarrow d \\ H_{n-*}(M) & \xleftarrow{u_*} & H_{n-*}(F_p) \end{array}$$

This means further that $(d^{-1} \circ u_* \circ d) \circ u^* = \pm id_{H^*(M)}$, hence u^* admits an inverse from the left. The algebraic reasoning made in the proof of Lemma 2.2.2 will furnish now the direct complement of $\text{Im } u^*$ in $H^*(F_p)$.

□

Remark 1. It is maybe worth to notice that, a subalgebra of a $\mathbb{Z}$ -algebra is not necessarily an additive direct summand: take for instance the subalgebra $\{k_0 + 2k_1x + \dots + 2k_nx^n : n \in \mathbb{N}, k_j \in \mathbb{Z}\}$ of the polynomial algebra $\mathbb{Z}[x]$. Of course, such problems cannot occur if instead of $\mathbb{Z}$, the field $\mathbb{Z}_2$ is involved.

We mentioned in the last section that $H^*(F_p) = \Delta(\xi_1, \dots, \xi_p)$ without going further in getting the complete ring structure. We fix again $r \in \{1, 2, \dots, p\}$ in the intention to obtain the decomposition of ξ_r^2 in terms of products of the form $\xi_{j_1}\xi_{j_2}$, with $j_1 < j_2$. As usual, our main instrument will be the sphere bundle $S^{m_{i_r}} \rightarrow F_r \xrightarrow{\pi_r} F_{r-1}$, together with its section $s_r : F_{r-1} \rightarrow F_r$. Also consider the involutive automorphism $\overline{\varphi}_r$ of F_r , which associates to an arbitrary $(z_1, \dots, z_r) \in F_r$ the element $(z_1, \dots, z_{r-1}, \varphi_{i_r}(z_r))$ of F_r . Notice that $\pi_r \circ \overline{\varphi}_r = \pi_r$. It follows that for any generator a_j^r of

$H_{lower}(F_r)$, $1 \leq j \leq r$, holds $\pi_{r*}\bar{\varphi}_{r*}(a_j^r) = \pi_{r*}(a_j^r)$, i.e. $\bar{\varphi}_{r*}(a_j^r) - a_j^r$ belongs to $\text{Ker}\pi_{r*}|_{H_{lower}(F_r)}$. By Lemma 2.2.3, the latter one is $\mathcal{R}a_r^r$ and we get in this way:

$$\bar{\varphi}_{r*}(a_j^r) = a_j^r + \beta_{jr}a_r^r,$$

for any $1 \leq j \leq r$, where β_{jr} are coefficients from $\mathcal{R}$. The following diagram is commutative:

$$\begin{array}{ccc} F_r & \xrightarrow{u_r} & M \\ \bar{\varphi}_r \downarrow & & \downarrow \varphi_{ir} \\ F_r & \xrightarrow{u_r} & M \end{array}$$

Hence from $u_{r*}(a_j^r) = b_{ij}$ it follows now $\varphi_{ir*}(b_{ij}) = b_{ij} + \beta_{jr}b_{ir}$. By comparing this relation to Thms. 2.1.1 and 2.1.2, we immediately identify the coefficients β_{jr} and get:

$$\bar{\varphi}_{r*}(a_j^r) = a_j^r - d_{ijir}a_r^r, \quad (2.12)$$

for any $1 \leq j \leq r$. Since the basis $\{\xi_j : 1 \leq j \leq r\}$ of $H^{lower}(F_r)$ is dual to the basis $\{a_j^r : 1 \leq j \leq r\}$ of $H_{lower}(F_r)$, a simple calculation leads to:

$$\bar{\varphi}_r^*(\xi_r) = (1 - d_{irir})\xi_r - \sum_{j=1}^{r-1} d_{ijir}\xi_j \quad (2.13)$$

The following technical result will be needed later:

Lemma 2.3.2 *Let $\pi : E \rightarrow S^1$ be a circle bundle over the circle S^1 and $s : S^1 \rightarrow E$ a section of it. Let $\bar{\varphi}$ denote the involutive automorphism of the total space E induced by the antipodal maps of the fibers. Let ξ_2 be the element of $H^1(E, \mathbb{Z}_2)$ uniquely determined by the fact that it generates, together with the unit, the cohomology of the fiber and it vanishes $s_*H_*(S^1, \mathbb{Z}_2)$. Then:*

$$\xi_2\bar{\varphi}^*(\xi_2) = 0.$$

PROOF One knows that there are only two bundles of the form described in our enunciation, namely the trivial one and the bundle induced by the Klein bottle (see for instance [36], §26).

In the first case, $E = S^1 \times S^1$, $\overline{\varphi}$ associates to any pair (z_1, z_2) from $S^1 \times S^1$ the pair $(z_1, -z_2)$, being homotopical to the identical map (we use the same argument as in Step 6. of the proof of Thm. 2.1.1). It follows $\xi_2 \overline{\varphi}^*(\xi_2) = \xi_2^2 = 0$, since this time ξ_2 is simply the generator of $H^1(S^1)$.

Consider now E the Klein bottle. It will be convenient for us to take the isoparametric submanifold of type A_2 with uniform multiplicity 1, together with the F-cycle F_2 associated to the gallery of type 12 and the projection $\pi = \pi_2 : F_2 = E \rightarrow F_1 + S^1$. In our previous notations, $\xi = \xi_2$ and $\overline{\varphi} = \overline{\varphi}_2$. It is clear that the bundle $\pi_2 : F_2 \rightarrow F_1$ is nontrivial, since from (2.12), $\overline{\varphi}_{2*}(a_2) = a_2 + a_1$. From (2.13) follows $\overline{\varphi}_2^*(\xi_2) = \xi_2 + \xi_1$. By using the description of the ring $H^*(E, \mathbb{Z}_2)$ (see for instance [28], Example 2, p.295), we can see that $\xi_2^2 = \xi_2 \xi_1$, and the lemma is completely proved. $\square$

The decisive step in obtaining the ring structure of $H^*(F_p)$ is:

Lemma 2.3.3 *It holds $\xi_r \overline{\varphi}_r^*(\xi_r) = 0$, for any $1 \leq r \leq p$.*

PROOF For any subset I of $\{1, 2, \dots, p\}$ which do not contain r holds $\xi_r \overline{\varphi}_r^*(\xi_r)(a_I) = \xi_r \overline{\varphi}_r^*(\xi_r)(s_{r*}(a_I)) = 0$, since $s_r^*(\xi_r) = 0$. It follows that we have the description:

$$\xi_r \overline{\varphi}_r^*(\xi_r) = \sum_{j=1}^{r-1} c_j \xi_j \xi_r. \quad (2.14)$$

We will show that all coefficients c_j involved in the formula above vanish, by taking separately the following situations:

Let's consider first the case when $m_{i_r} \geq 2$. Take $J = \{1 \leq j \leq r : m_{i_j} = m_{i_r}\}$ and $J' = J \setminus \{r\}$. Then the restriction map

$$\pi_r : a_J \rightarrow a_{J'} \quad (2.15)$$

is also a $S^{m_{i_r}}$ -bundle, which is clearly orientable (since $a_{J'}$ is simply connected). The restriction

$$s_r : a_{J'} \rightarrow a_J \quad (2.16)$$

is a section of this bundle. If $\iota : a_J \rightarrow F_p$ is the inclusion map, having in mind equation (2.14), it will be enough to show that $\iota^*(\xi_r \overline{\varphi}_r^*(\xi_r)) = (\iota^*(\xi_r)) \overline{\varphi}_r^*(\iota^*(\xi_r)) = 0$. To this end, we will restrict our attention to the oriented $S^{m_{i_r}}$ -bundle given by (2.15), with the section described in (2.16) and

the involutive automorphism $\overline{\varphi}_r$ on a_J . Associate to this bundle the Gysin sequence, which has the form:

$$\dots \rightarrow H^q(a_{J'}) \xrightarrow{\pi_r^*} H^q(a_J) \xrightarrow{\tau} H^{q-m_{i_r}}(a_{J'}) \rightarrow H^{q+1}(a_{J'}) \rightarrow \dots$$

By using Thm.21.9 of [24], $\pi_r^* \tau((i^* \xi_r) \overline{\varphi}_r^*(i^* \xi_r)) = 0$. But in our case, π_r^* is injective, hence there exists $u \in H^*(a_{J'})$ with $(i^* \xi_r) \overline{\varphi}_r^*(i^* \xi_r) = \pi_r^*(u)$. By applying s_r^* to both terms of the last equality, one gets $u = 0$, since $s_r^*(i^* \xi_r) = 0$ and $\pi_r \circ s_r = id$.

When $m_{i_r} = 1$, we have only to take an arbitrary $1 \leq j < r$ with $m_{i_j} = 1$ and to notice that if $j : a_{\{j,r\}} \rightarrow F_r$ is the inclusion map, then $\xi_r \overline{\varphi}_r^*(\xi_r)(a_{\{j,r\}}) = \xi_r \overline{\varphi}_r^*(\xi_r)(j_*(a_{\{j,r\}})) = (j^* \xi_r) \overline{\varphi}_r^*(j^* \xi_r)(a_{\{j,r\}}) = 0$, because of Lemma 2.3.2.

□

We are now in position to state:

Proposition 2.3.4 *The algebra $H^*(F_p)$ is simply generated by $\xi_1, \xi_2, \dots, \xi_p$, where $\deg \xi_r = m_{i_r}$ and :*

$$(1 - d_{i_r i_r}) \xi_r^2 = \sum_{j=1}^{r-1} d_{i_j i_r} \xi_r \xi_j,$$

for all $1 \leq r \leq p$.

We know now that, via u_* , $H^*(M)$ becomes a subring of $H^*(F_p)$. Fix $\mathcal{G}_m^0$ a set of representatives of $\mathcal{G}_m / \sim$. Recall that we stated in Thm.1.5.9 a natural identification of $H_*(M)$ with the free $\mathcal{R}$ -module $[\mathcal{G}_m^0]$. Consider now the dual basis $\{\omega_f : f \in \mathcal{G}_m^0\}$ of $H^*(M)$, i.e.:

$$\omega_f(g) = \begin{cases} 1, & \text{if } f = g \\ 0, & \text{otherwise.} \end{cases}$$

Since the ring $H^*(F_p)$ is already known, it only remains to express an arbitrary ω_f , $f \in \mathcal{G}_m^0$, in terms of the basis $\{\xi_J : J \subseteq \mathcal{I}_p\}$ of $H^*(F_p)$. In order to do that, it will be enough to estimate the value of $u^*(\omega_f)([a_J])$, for any $J \subseteq \mathcal{I}_p$. If J is of the form $\{j_1 < j_2 < \dots, j_r\}$, we associate the gallery $g(J) \in \mathcal{G}_m$ uniquely determined by its type $i_{j_1} i_{j_2} \dots i_{j_r}$ and then the F-cycle $(F_{g(J)}, u_{g(J)})$ (see section 1.5). Consider the degree +1 diffeomorphism

$$\begin{array}{ccc}
a_J & \xrightarrow{\psi_J} & F_{g(J)} \\
& \searrow u & \nearrow u_{g(J)} \\
& M &
\end{array}$$

$\psi_J : a_J \rightarrow F_{g(J)}$ which maps an arbitrary $(y_1, \dots, y_p) \in a_J$ to $(y_{j_1}, \dots, y_{j_r})$ and makes the following diagram commutative:

It follows from here that $u_*([a_J]) \neq 0$ iff $g(J) \in \mathcal{G}_m$. Whenever this latter condition is satisfied, we take f the element of $\mathcal{G}_m^0$ homotopic to $g(J)$ and denote $s(J) = s(f, g(J))$ (see (1.10)); it holds then by Prop.1.5.7:

$$u_*([a_J]) = u_{g(J)*}[F_{g(J)}] = (-1)^{s(J)} f. \quad (2.17)$$

Hence we have for any $f \in \mathcal{G}_m^0$:

$$u^*(\omega_f)([a_J]) = \omega_f(u_*[a_J]) =$$

$$= \omega_f(u_{g(J)*}[F_{g(J)}]) = \begin{cases} (-1)^{s(J)}, & \text{if } g(J) \in \mathcal{G}_m \text{ and } g(J) \sim f \\ 0, & \text{otherwise.} \end{cases}$$

From Propositions 2.2.4 and 2.2.5 follows now the exact form of ω_f . We conclude by:

Theorem 2.3.5 *The $\mathcal{R}$ -algebra $H^*(M)$ is the subalgebra of $H^*(F_p)$ which is additively freely generated by the set of elements of the form:*

$$\omega_f = \Sigma\{(-1)^{l(J)+s(J)}\xi_J : J \subseteq \mathcal{I}_p \text{ with } g(J) \sim f\}, \quad (2.18)$$

for all $f \in \mathcal{G}_m^0$.

Let us consider now v a parallel normal field on M and associate to it the focal manifold M_v , as image of the end-point map π_v . In the same way as we did in section 1.6, we associate to v the subgroup W_v and the system of minimal representatives W^v . If we combine Thm.1.5.9 and Thm.1.6.9 we immediately obtain:

Proposition 2.3.6 *The subset $\{f : s_f \in W^v\}$ of $\mathcal{G}_m^0$ is mapped by π_{v*} bijectively onto a basis of $H_*(M_v)$. If $f \in \mathcal{G}_m^0$ such that $s_f \notin W^v$, then $\pi_{v*}f = 0$.*

Whenever $\mathcal{R} = \mathbb{Z}$ is possible, no torsion occurs in homology, neither for M , nor for M_v . Hence $H^*(M) = \text{Hom}(H_*(M), \mathbb{Z})$, $H^*(M_v) = \text{Hom}(H_*(M_v), \mathbb{Z})$ and since π_{v*} is surjective, it follows that $\pi_v^* : H^*(M_v) \rightarrow H^*(M)$ is injective. The same argument which we used in order to prove Thm.2.2.4 will give now:

Theorem 2.3.7 *The $\mathcal{R}$ -algebra $H^*(M_v)$ is the subalgebra of $H^*(F_p)$ additively freely generated by those elements ω_f , with $f \in \mathcal{G}_m^0$, for which s_f is in W^v .*

Remark 2. As we are expecting at, the set of generators of $H^*(M_v)$ in $H^*(F_p)$ described by (2.15) is independent of the choice of representatives in $\mathcal{G}_m^0 \subseteq \mathcal{G}_m$. This is quite clear when only the ring $\mathcal{R} = \mathbb{Z}_2$ is involved. In order to see that also when $\mathcal{R} = \mathbb{Z}$ is allowed, notice first that whenever f, g and h are types of homotopical minimal galleries from $\mathcal{G}_m$, holds $s(f, g) + s(g, h) + s(h, f) \equiv 0 \pmod{2}$. Consequently, if f and f' are minimal and homotopic, $f' \sim f$, then ω_f differ of $\omega_{f'}$ only by the sign $(-1)^{s(f, f')}$.

Remark 3. The last two results furnish the $\mathcal{R}$ -cohomology ring of any element of an irreducible isoparametric foliation in terms of the root system of the corresponding symmetric space, the multiplicities and the position of the "initial" focalizing vector in the fundamental Weyl chamber (in other words, in terms of the marked Dynkin diagram, the nodes where "one focalizes"). This is what we could call the quantitative side of these estimations. But it is not easy to deduce from here usual algebraic descriptions of $H^*(M_v)$ in terms of generators and relations: for instance, we will be interested in our next chapter about the $\mathcal{R}$ -cohomological length of M_v and there are not too many chances to calculate that directly from the descriptions above. Very helpful in this intention is the other side that we would like to point out in this context, namely the qualitative one. It will constitute the subject of the next statement. The arguments which support it are clear, but it is maybe worse to point out what happens when $\mathcal{R} = \mathbb{Z}$: when we have uniform, even multiplicity, the exact values of $l(J)$ and $s(J)$ does not matter, since they are even; when the multiplicities are different and both greater than 1, namely in the $(BC)_k IV$ and $C_k IV$ situations, we have the marked Coxeter graph of the form:

$$\begin{array}{ccccccc} m_1 & m_1 & m_1 & m_1 & m_2 \\ \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$$

where m_1 and m_2 are greater than 1, m_1 always even and m_2 always odd. In order to have the certitude that m_1 and m_2 are also relatively prime, we will suppose $k \geq 3$ (the case $k = 2$ will be however analysed separately in the next section).

Corollary 2.3.8 *If M is a homogeneous isoparametric submanifold of rank at least 3, then the ring $H^*(M)$ is uniquely determined, up to a shifting of the dimensions of generators, by the first coloured Dynkin diagram, and the choice of a minimal gallery of maximal length in W . The same data, together with the subset $I(v)$ of the set of nodes, furnish in a unique manner the $\mathcal{R}$ -cohomology ring of M_v .*

We reduced in this way the calculation of $H^*(M_v)$ to a very few number of cases. One of the main goals of our paper (unfortunately, only incomplete achieved) is to obtain the $\mathcal{R}$ -cohomology at least for the isoparametric elements of the foliations, i.e. $H^*(M)$. We will take one by one the 17 first coloured Dynkin diagrams and find for each of them a reasonable representative, which will say everything about all other elements of its class.

As can be easily seen, in the case of a rank two Weyl group, two minimal galleries with the same start-point which are homotopic, must be necessarily equal, provided they are not of maximal length: if J and J' are subsets of $\mathcal{I}_p$ with $g(J) \sim g(J')$ minimal of non-maximal rank, then $g(J) = g(J')$, hence $s(J) = s(J')$ and $l(J) = l(J')$. We have proved:

Corollary 2.3.9 *Let M be a rank two isoparametric submanifold.*

(i) *The cohomology algebra of M is the subalgebra of $H^*(F_p)$ (see Prop. 2.3.3) which is additively freely generated by the elements:*

$$\omega_f = \Sigma\{\xi_J : J \subseteq \mathcal{I}_p \text{ with } g(J) \sim f\},$$

$f \in \mathcal{G}_m$.

(ii) *If v is a parallel normal field, then the cohomology algebra of the corresponding focal manifold M_v is the subalgebra of $H^*(F_p)$ which is freely generated by those elements ω_f described above for which $s_f \in W^v$.*

We will come back and use this statement in our next section, in order to calculate the cohomology ring of all isoparametric hypersurfaces in spheres. But for the moment we would like to prove two main results of Bott and

Samelson's paper [5], actually slightly generalized, by our own methods. The intention is to describe the integral cohomology of the elements of an irreducible isoparametric foliation with uniform, even multiplicity. Concretely, the spaces involved here are the adjoint orbits of compact simple Lie groups, the flag manifolds over the quaternion algebra $\mathbf{H}$ and Cayley algebra $\mathbf{O}$, the isoparametric submanifold of type G_2 with multiplicity 2 and its two focal manifolds (as we already noticed, the fact that the latter one is also an adjoint orbit is, for the moment, only conjectured). Directly inspired by Thm.III of [5], we state for these spaces:

Corollary 2.3.10 *Let M be an isoparametric submanifold of uniform even multiplicity. Then, via u^* , $H^*(M, \mathbb{Z})$ is the smallest subalgebra of $H^*(F_p, \mathbb{Z})$ with the following two properties:*

- (1) *it contains the unit and all elements of the form $\omega_i = \Sigma\{\xi_j : i_j = i\}$, with $1 \leq i \leq k$;*
- (2) *it is additively a direct summand.*

PROOF Let us denote by A the smallest subalgebra of $H^*(F_p, \mathbb{Z})$ which satisfies (1) and (2). By Lemma 2.3.1 and Thm. 2.3.5, it is clear that all we have to show is $H^*(M, \mathbb{Z}_2) \subseteq A$.

Consider first the case when M is a principal adjoint orbit. By a classical result of Borel we know that $H^*(M, \mathbb{Q})$ is generated by 1 and $H^2(M, \mathbb{Q})$. This means that if ω is an arbitrary element of $H^*(M, \mathbb{Z})$, then we can find an integer p with $p\omega \in A$. Since the same ω can be decomposed as $\omega_1 + \omega_2$, with $\omega_1 \in A$ and ω_2 in its complement, it follows $p\omega_2 = 0$, hence $\omega_2 = 0$, (there are no torsioned elements in $H^*(F_p)$).

If M is not any more an adjoint orbit, consider G a connected and compact Lie group whose Weyl group is the same as of M and $M' = G/T$ a principal isotropy orbit (T denotes a maximal torus in G). To the same minimal gallery of maximal length in W correspond in M and M' the cycles of highest dimension F_p , respectively F'_p . Up to an eventual shifting of the dimensions of generators ξ_i , $1 \leq i \leq p$, the cohomology rings of F_p and F'_p are the same, over $\mathbb{Z}$ and implicitly over $\mathbb{Q}$. Since the elements ω'_i , $1 \leq i \leq k$ generates, together with the unit 1, the rational cohomology of M' , it follows that the corresponding elements ω_i generate, together with 1, the algebra $H^*(M, \mathbb{Q})$. We conclude by applying once again the same argument as in the previous case.

□

Once again inspired by Bott and Samelson, this time by their Thm.III' of [5], we describe the cohomology algebra of an arbitrary focal manifold, in the same context as before. We must notice that this time, our proof is totally different of their proof: instead of their spectral sequences argument (which actually works only for adjoint orbits), we founded a purely geometric one.

Corollary 2.3.11 *Under the same assumptions as in the previous statement, we consider v a normal parallel field on M and associate the focal manifold M_v . Then $H^*(M_v, \mathbb{Z})$ is the subalgebra of $H^*(M, \mathbb{Z})$ consisting of the W_v -invariant elements.*

PROOF We have already explained why the map $\pi_v^* : H^*(M_v) \rightarrow H^*(M)$ is injective; we identify $H^*(M_v)$ to the image of this map. For any $i \in I(v)$ holds $\varphi_i \circ \pi_v = \pi_v$, hence any element of $H^*(M_v)$ is W_v -invariant.

Suppose now that there exists a linear combination of the form $\omega = \sum c_g \omega_g$, where $g \in \mathcal{G}_m^0$, $c_g \in \mathbb{Z} \setminus \{0\}$ and $s_g \notin W^v$ such that $\varphi_i^*(\omega) = \omega$ for all $i \in I(v)$. Fix a certain f among the elements of $\mathcal{G}_m^0$ involved in the sum from above, together with the corresponding F-cycle (F_f, u_f) . Since for an arbitrary $g \in \mathcal{G}_m^0$ holds:

$$u_f^*(\omega_g)[F_f] = \omega_g(f) = \begin{cases} 1, & \text{if } f = g \\ 0, & \text{otherwise,} \end{cases}$$

it follows that $u_f^*(\omega) = c_f u_f^*(\omega_f)$. The fact that $s_f \notin W^v$ means that the word f is of the form $f = f' i_0$, with f' the type of a minimal gallery and $i_0 \in I(v)$. Let $\bar{\varphi}_{i_0}$ be the involutive automorphism of F_f which associates to an arbitrary $(z_1, \dots, z_r) \in F_f$ the element $(z_1, \dots, z_{r-1}, \varphi_{i_0}(z_r))$, the latter one being clearly also in F_f . From the commutativity of the diagram below

$$\begin{array}{ccc} F_f & \xrightarrow{u_f} & M \\ \bar{\varphi}_{i_0} \downarrow & & \downarrow \varphi_{i_0} \\ F_f & \xrightarrow{u_f} & M \end{array}$$

follows $\bar{\varphi}_{i_0}^* u_f^*(\omega) = u_f^*(\omega)$, hence

$$\bar{\varphi}_{i_0}^* u_f^*(\omega_f) = u_f^*(\omega_f).$$

Notice that $u_f^*(\omega_f)$ generates the top-cohomology of F_f . The map $p : F_f \rightarrow F_{f'}$, $(z_1, \dots, z_r) \mapsto (z_1, \dots, z_{r-1}), (z_1, \dots, z_r) \in F_f$, is an $S^{m_{i_0}}$ -sphere bundle for which $s : F_{f'} \rightarrow F_f$, $(z_1, \dots, z_{r-1}) \mapsto (z_1, \dots, z_{r-1}, z_{r-1}), (z_1, \dots, z_{r-1}) \in F_{f'}$, is a section. The same reasoning as in Proposition 2.2.1 shows that $H_{m_{i_0}}(F_f)$ is generated by the fundamental homology classes of all m_{i_0} -dimensional fundamental curvature spheres whose indices are involved in the word f . If $a \in H_{m_{i_0}}(F_f)$ generates the top-homology of the fiber for the bundle above, take $\xi \in H^{m_{i_0}}(F_f)$ uniquely determined by $\xi(a) = 1$ and the fact it vanishes the fundamental homology classes of all other fundamental curvature spheres regarded in $H_{m_{i_0}}(F_f)$. It follows that the restriction of ξ to any fiber generates, together with the unit, the cohomology of this fiber. By the theorem of Leray-Hirsch, the map p^* is injective and $H^*(F_p) = p^*(H^*(F_{f'})) \oplus \xi p^*(H^*(F_{f'}))$. In particular, if θ generates the top-cohomology of $F_{f'}$, then $\xi p^*(\theta)$ is a generator of the top-cohomology of F_f . Hence $\overline{\varphi}_{i_0}^*(\xi p^*(\theta)) = \xi p^*(\theta)$. Since m_{i_0} is even, $\overline{\varphi}_{i_0*}(a) = -a$. Consequently, $\varphi_{i_0}^*(\xi)$ must be of the form $-\xi + p^*(\psi)$, $\psi \in H^*(F_{f'})$. But then $\overline{\varphi}_{i_0}^*(\xi p^*(\theta)) = -\xi p^*(\theta) + p^*(\theta\psi)$, and the latter sum can be $\xi p^*(\theta)$ only if $\xi p^*(\theta) = 0$. That is clearly impossible. □

In the remaining part of the section we will restrict our attention exclusively to the coefficient ring $\mathbb{Z}_2$. It makes no sense any more to fix the reduced expressions for all elements of W (recall that their only use was to distinguish the signs $(-1)^{s(J)}$).

Theorem 2.3.12 *Let M be an arbitrary isoparametric submanifold endowed with a minimal gallery of maximal length. Denote by $i_1 \dots i_p$ the type of this gallery and by F_p the corresponding F -cycle.*

a) *The $\mathbb{Z}_2$ -cohomology of F_p is given by:*

$$H^*(F_p, \mathbb{Z}_2) = \mathbb{Z}_2[\xi_1, \dots, \xi_p]/I,$$

where $\deg \xi_j = m_{i_j}$, $1 \leq j \leq p$ and I is the ideal generated by $\{\xi_r^2 + \sum_{j=1}^{r-1} d'_{i_j i_r} \xi_r \xi_j : 1 \leq r \leq p\}$, the coefficients d'_{kl} being described by (2.4).

b) *The algebra $H^*(M, \mathbb{Z}_2)$ is the subalgebra of $H^*(F_p, \mathbb{Z}_2)$ which is additively freely generated by all elements of the form:*

$$\Sigma\{\xi_J : J \subseteq \mathcal{I}_p \text{ with } g(J) \sim f\},$$

with $f \in \mathcal{G}_m$.

c) If v is a parallel normal vector field on M , then the $\mathbb{Z}_2$ -cohomology algebra of the focal manifold M_v is the subalgebra of $H^*(F_p)$ which is additively freely generated by all elements described at the point b) with $s_f \in W^v$.

d) If M is a homogeneous isoparametric submanifold of rank greater than 2, then the multiplicities m_1 and m_2 are equal or relatively prime. The algebra $H^*(M, \mathbb{Z}_2)$ is uniquely determined, up to a shifting of the dimensions of generators, by the second coloured Dynkin diagram of the corresponding symmetric space and the choice of the minimal gallery of maximal length in W . The same data, together with the subset $I(v)$ of the set of nodes, determine uniquely the cohomology ring of the focal manifold M_v .

The last statement will be later our basic tool in "solving" the case of uniform multiplicity 1 via uniform multiplicity 2 (i.e. adjoint orbits) and then the generic case $C_k III$ via $C_k IV$ (see sections 2.5 and 2.6 below). But for the moment we will exemplify the general methods exposed in this section in order to obtain the $\mathcal{R}$ -cohomology ring of all isoparametric hypersurfaces in the sphere and their focal manifolds.

2.4 Isoparametric hypersurfaces in spheres

Talking about isoparametric hypersurfaces in spheres or about rank 2 isoparametric submanifolds in Euclidean spaces is the same (cf. [32], Prop.6.3.17). We would like now to use this well-known context (see for instance [29], [30], [1]), in order to test here the general methods of the previous section. Our basic tool will be, of course, Cor.2.3.9. After a case by case analysis, the $\mathcal{R}$ -cohomology of the isoparametric submanifold M and of its both focal manifolds M_1 and M_2 will be obtained. We have in this case only two focal manifolds, say them M_1 and M_2 , with the indices chosen in such a way, that the end-point map $\pi_i : M \rightarrow M_i$ an S^{m_i} -bundle, for each $i \in \{1, 2\}$.

1. M is of A_2 type.

We consider the minimal gallery whose type is $f_0 = 121$ (see Figure 2.4 below) and associate the F-cycle

$$F_{f_0} = F_3 = \{(z_1, z_2, z_3) \in M^{\times 3} : z_1 \in S_1(x_0), z_2 \in S_2(z_1), z_3 \in S_1(z_2)\}.$$

Distinguish the following situations:

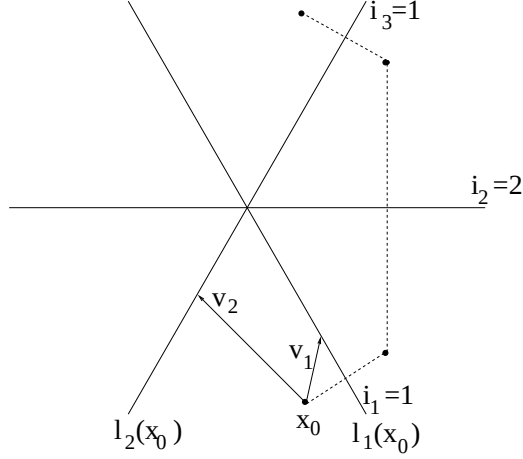


Figure 2.4: The case A_2 .

1a) $m_1 = m_2 \stackrel{not.}{=} m \in \{2, 4, 8\}$

It follows by Prop.2.3.4 and Thm.2.1.2, b), that:

$$H^*(F_3, \mathbb{Z}) = \mathbb{Z}[\xi_1, \xi_2, \xi_3]/I$$

where I is the ideal generated by $\xi_3^2 - \xi_3\xi_2 + 2\xi_3\xi_1$, $\xi_2^2 - \xi_2\xi_1$ and ξ_1^2 . By Cor.2.3.9, $H^*(M, \mathbb{Z})$ is the subalgebra of $H^*(F_3, \mathbb{Z})$ additively generated by the following elements: 1 , $\omega_1 = \xi_1 + \xi_3$, $\omega_2 = \xi_2$, $\omega_{12} = \xi_1\xi_2$, $\omega_{21} = \xi_2\xi_3$ and $\omega_{121} = \xi_1\xi_2\xi_3$. A few elementary calculations lead to:

$$H^*(M, \mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_2]/(\omega_1^2 + \omega_2^2 - \omega_1\omega_2, \omega_1^3, \omega_2^3), \quad (2.19)$$

with $\deg \omega_1 = \deg \omega_2 = m$. We can also use the variables: $\eta_1 = \omega_1$, $\eta_2 = -\omega_2$, $\eta_3 = \omega_2 - \omega_1$ in order to obtain an expected expression (cf. section 2.5, point 1.):

$$H^*(M, \mathbb{Z}) = \mathbb{Z}[\eta_1, \eta_2, \eta_3]/(\eta_1 + \eta_2 + \eta_3, \eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3, \eta_1\eta_2\eta_3). \quad (2.20)$$

In order to calculate $H^*(M_1)$, consider $W_{v_1} = \{1, s_1\}$ and consequently $W^{v_1} = \{1, s_2, s_1s_2\}$. The point (ii) of the same Cor. 2.3.8 furnishes $H^*(M_1, \mathbb{Z})$ as the submodule generated by 1 , ω_2 and ω_{12} inside $H^*(F_p, \mathbb{Z})$. This means concretely:

$$H^*(M_1, \mathbb{Z}) = \mathbb{Z}[\omega_2]/(\omega_2^3) \quad (2.21)$$

and similarly:

$$H^*(M_2, \mathbb{Z}) = \mathbb{Z}[\omega_1]/(\omega_1^3). \quad (2.22)$$

1b) $m_1 = m_2 = 1$.

We apply Thm.2.3.12 and use the informations already obtained at the previous point in order to obtain:

$$H^*(M, \mathbb{Z}_2) = \mathbb{Z}_2[\eta_1, \eta_2, \eta_3]/(\eta_1 + \eta_2 + \eta_3, \eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3, \eta_1\eta_2\eta_3), \quad (2.23)$$

as well as

$$H^*(M_1, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_2]/(\omega_2^3) \quad (2.24)$$

and

$$H^*(M_2, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1]/(\omega_1^3), \quad (2.25)$$

the dimension of the generators being this time 1.

2. M is of B_2 type.

We associate this time to $f_0 = 1212$ (see Figure 2.5) the F-cycle

$$F_4 = \{(z_1, z_2, z_3, z_4) : z_1 \in S_1(x_0), z_2 \in S_2(z_1), z_3 \in S_1(z_2), z_4 \in S_2(z_3)\}$$

2a) $m_1 = m_2 = 2$.

The coefficients d_{ij} are given this time by the Cartan matrix:

$$(d_{ij}) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

and we obtain in this way

$$H^*(F_4, \mathbb{Z}) = \mathbb{Z}[\xi_1, \xi_2, \xi_3, \xi_4]/I$$

with the ideal I generated by $\xi_4^2 - \xi_4\xi_3 + 2\xi_4\xi_2 - \xi_4\xi_1$, $\xi_3^2 - 2\xi_3\xi_2 + 2\xi_3\xi_1$, $\xi_2^2 - \xi_2\xi_1$, ξ_1^2 . Additively, $H^*(M, \mathbb{Z})$ is generated by 1, $\omega_1 = \xi_1 + \xi_3$, $\omega_2 = \xi_2 + \xi_4$, $\omega_{12} = \xi_1\xi_2 + \xi_1\xi_4 + \xi_3\xi_4$, $\omega_{21} = \xi_2\xi_3$, $\omega_{121} = \xi_1\xi_2\xi_3$, $\omega_{212} = \xi_2\xi_3\xi_4$ and $\omega_{1212} = \xi_1\xi_2\xi_3\xi_4$. After some calculations we get:

$$H^*(M, \mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_2]/((\omega_1 - \omega_2)^2 + \omega_2^2, (\omega_1 - \omega_2)^2\omega_2^2),$$

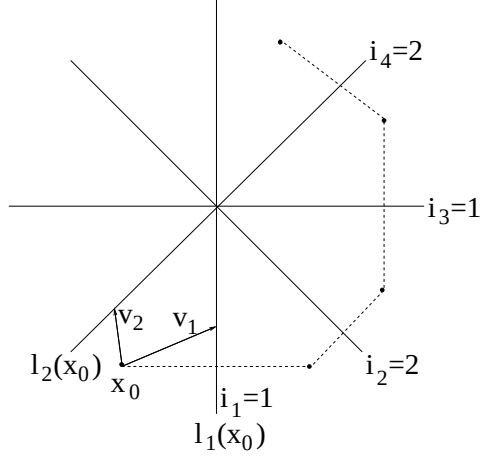


Figure 2.5: The case B_2 .

where ω_1 and ω_2 are both of degree 2.

To $W_{v_1} = \{1, s_1\}$ we associate $W^{v_1} = \{1, s_2, s_1 s_2, s_2 s_1 s_2\}$ and obtain in this way $H^*(M_1, \mathbb{Z})$ as the subalgebra of $H^*(F_4, \mathbb{Z})$ which is additively generated by 1, ω_2 , ω_{12} , ω_{212} . More precisely:

$$H^*(M_1, \mathbb{Z}) = \mathbb{Z}[\omega_2]/(\omega_2^4). \quad (2.26)$$

By a similar method, the cohomology of M_2 will be obtained as:

$$H^*(M_2, \mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_{21}, \omega_{121}]/(\omega_1^2 - 2\omega_{21}, \omega_1^3 - 4\omega_{121}, \omega_1^4), \quad (2.27)$$

with $\deg \omega_1 = 2$, $\deg \omega_{21} = 4$ and $\deg \omega_{121} = 6$. Notice that, even if the principal adjoint orbit M has the integral cohomology generated as ring by 1 and its elements of degree 2, not the same can be said about the subprincipal orbit M_2 . Somethig like that cannot happen any more if the rational cohomology is involved. It follows indeed from (2.23) that:

$$H^*(M_2, \mathbb{Q}) = \mathbb{Q}[\omega_1]/(\omega_1^4). \quad (2.28)$$

2b) $m_1 = m_2 = 1$.

By Thm.2.3.11, we only have to take the results from the previous step, to tensorize with $\mathbb{Z}_2$ and to shift the degrees of the generators from 2 to 1.

One obtains:

$$H^*(M, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \omega_2]/(\omega_1^2, \omega_2^4) \quad (2.29)$$

$$H^*(M_1, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_2]/(\omega_2^4) \quad (2.30)$$

$$H^*(M_2, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \omega_{21}, \omega_{121}]/(\omega_1^2, \omega_1\omega_{21}, \omega_1\omega_{121}, \omega_{21}^2, \omega_{121}^2), \quad (2.31)$$

with $\deg\omega_1 = \deg\omega_2 = 1$, $\deg\omega_{21} = 2$, $\deg\omega_{121} = 3$.

2c) $m_1 \neq m_2$, both greater than 1.

One knows that one of the multiplicities, say it m_1 , is even and the other is odd (this follows easily from the main result of [1]). This time we have:

$$(d_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

and consequently

$$H^*(F_4, \mathbb{Z}) = \Delta(\xi_1, \xi_2, \xi_3, \xi_4),$$

$\deg\xi_1 = \deg\xi_3 = m_1$ and $\deg\xi_2 = \deg\xi_4 = m_2$, with the relations: $\xi_4^2 = 0$, $\xi_3^2 + 2\xi_3\xi_1 = 0$, $\xi_2^2 = 0$, $\xi_1^2 = 0$. We can see $H^*(M, \mathbb{Z})$ inside $H^*(F_4, \mathbb{Z})$ as generated additively by the same elements $\omega_1, \dots, \omega_{1212}$ as at the previous point. Hence

$$H^*(M, \mathbb{Z}) = \Lambda(\omega_1, \omega_2, \omega_{12}), \quad (2.32)$$

with $\deg\omega_1 = m_1$, $\deg\omega_2 = m_2$ and $\deg\omega_{12} = m_1 + m_2$. Once again $H^*(M_1, \mathbb{Z})$ is generated by 1, ω_2 , ω_{12} and ω_{212} , hence:

$$H^*(M_1, \mathbb{Z}) = \Lambda(\omega_2, \omega_{12}), \quad (2.33)$$

with $\deg\omega_2 = m_2$, $\deg\omega_{12} = m_1 + m_2$ and $H^*(M_2, \mathbb{Z})$ is generated by 1, ω_1 , ω_{21} and ω_{121} , i.e.:

$$H^*(M_2, \mathbb{Z}) = \Lambda(\omega_1, \omega_{21}), \quad (2.34)$$

with $\deg\omega_1 = m_1$, $\deg\omega_{21} = m_1 + m_2$.

2d) $1 = m_1 < m_2$.

This time $\mathcal{R} = \mathbb{Z}_2$ and all d_{ij} must be 0. It follows:

$$H^*(F_4, \mathbb{Z}_2) = \Lambda(\xi_1, \xi_2, \xi_3, \xi_4)$$

with $\deg\xi_1 = \deg\xi_3 = m_1$, $\deg\xi_2 = \deg\xi_4 = m_2$,

$$H^*(M, \mathbb{Z}_2) = \Lambda(\omega_1, \omega_2, \omega_{12}) \quad (2.35)$$

$$H^*(M_1, \mathbb{Z}_2) = \Lambda(\omega_2, \omega_{12}) \quad (2.36)$$

$$H^*(M_2, \mathbb{Z}_2) = \Lambda(\omega_1, \omega_{21}), \quad (2.37)$$

with $\deg \omega_1 = 1$, $\deg \omega_2 = m_2$, $\deg \omega_{12} = \deg \omega_{21} = 1 + m_2$.

3. M is of G_2 type.

Take $f_0 = 121212$ as in Figure 2.6 and associate to it

$$F_6 = \{(z_1, \dots, z_6) : z_1 \in S_1(x_0), z_2 \in S_2(z_1), z_3 \in S_1(z_2), z_4 \in S_2(z_3), z_5 \in S_1(z_4), z_6 \in S_2(z_5)\}.$$

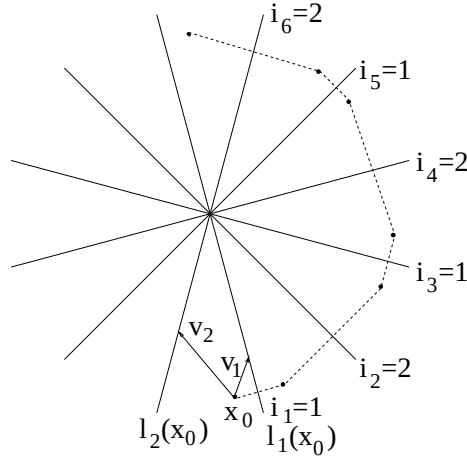


Figure 2.6: The case G_2 .

3a) $m_1 = m_2 = 2$.

We have

$$(d_{ij}) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

hence

$$H^*(F_6, \mathbb{Z}) = \mathbb{Z}[\xi_1, \dots, \xi_6]/I$$

where I is the ideal generated by : $\xi_6^2 - 3\xi_6\xi_5 + 2\xi_6\xi_4 - 3\xi_6\xi_3 + 2\xi_6\xi_2 - 3\xi_6\xi_1$, $\xi_5^2 - \xi_5\xi_4 + 2\xi_5\xi_3 - \xi_5\xi_2 + 2\xi_5\xi_1$, $\xi_4^2 - 3\xi_4\xi_3 + 2\xi_4\xi_2 - 3\xi_4\xi_1$, $\xi_3^2 - \xi_3\xi_2 + 2\xi_3\xi_1$, $\xi_2^2 - 3\xi_2\xi_1$ and ξ_1^2 and $H^*(M, \mathbb{Z})$ is the subring additively freely generated by:

$$\omega_1 = \xi_1 + \xi_3 + \xi_5,$$

$$\omega_2 = \xi_2 + \xi_4 + \xi_6,$$

$$\begin{aligned}
\omega_{12} &= \xi_1\xi_2 + \xi_1\xi_4 + \xi_1\xi_6 + \xi_3\xi_4 + \xi_3\xi_6 + \xi_5\xi_6, \\
\omega_{21} &= \xi_2\xi_3 + \xi_2\xi_5 + \xi_4\xi_5, \\
\omega_{121} &= \xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_5 + \xi_1\xi_4\xi_5 + \xi_3\xi_4\xi_5, \\
\omega_{212} &= \xi_2\xi_3\xi_4 + \xi_2\xi_5\xi_6 + \xi_4\xi_5\xi_6 + \xi_2\xi_3\xi_6, \\
\omega_{1212} &= \xi_1\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_3\xi_6 + \xi_1\xi_4\xi_5\xi_6 + \xi_3\xi_4\xi_5\xi_6 + \xi_1\xi_2\xi_5\xi_6, \\
\omega_{2121} &= \xi_2\xi_3\xi_4\xi_5, \\
\omega_{12121} &= \xi_1\xi_2\xi_3\xi_4\xi_5, \\
\omega_{21212} &= \xi_2\xi_3\xi_4\xi_5\xi_6, \\
\omega_{121212} &= \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6.
\end{aligned}$$

After some calculations we recover the result stated in Thm.III' of [5] (even if it is still possible to have a such M which is not the principal isotropy orbit of the connected Lie group of G_2 type, this M must have the same integral cohomology as the up-mentioned orbit): $H^*(M, \mathbb{Z})$ has an additive basis consisting of $1, \omega_1, \omega_2, \omega_1^2, \omega_1\omega_2, \omega_1^2\omega_2, 1/2\omega_1^3, 1/2\omega_1^4, 1/2\omega_1^3\omega_2, 1/2\omega_1^5, 1/2\omega_1^4\omega_2, 1/2\omega_1^5\omega_2$, where ω_1, ω_2 are subjects of the only relations : $3\omega_1^2 + \omega_2^2 - 3\omega_1\omega_2 = 0, \omega_1^6 = \omega_2^6 = 0$. We can also express $H^*(M, \mathbb{Z})$ in the same manner as since now:

$$H^*(M, \mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_2, \eta] / (3\omega_1^2 + \omega_2^2 - 3\omega_1\omega_2, 2\eta - \omega_1^3, \omega_1^6, \omega_2^6), \quad (2.38)$$

with $\deg\omega_1 = \deg\omega_2 = 2, \deg\eta = 6$. We will use this particular situation to convince ourselves once more that the situation becomes more simple if the coefficient ring is $\mathbb{Q}$:

$$H^*(M, \mathbb{Q}) = \mathbb{Q}[\omega_1, \omega_2] / (3\omega_1^2 + \omega_2^2 - 3\omega_1\omega_2, \omega_1^6, \omega_2^6). \quad (2.39)$$

As about the cohomology of M_1 , notice that $W_{v_1} = \{1, s_1\}$, $W^{v_1} = \{1, s_2, s_1s_2, s_2s_1s_2, s_1s_2s_1s_2, s_2s_1s_2s_1s_2\}$ and conclude that $H^*(M_1, \mathbb{Z})$ is freely generated by $1, \omega_2, \omega_{12}, \omega_{212}, \omega_{1212}$ and ω_{21212} , i.e. by $1, \omega_2, 1/3\omega_2^2, 1/6\omega_2^3, 1/18\omega_2^4, 1/18\omega_2^5$. Similarly, $H^*(M_2, \mathbb{Z})$ has an additive basis consisting of $1, \omega_1, \omega_1^2, 1/2\omega_1^3, 1/2\omega_1^4, 1/2\omega_1^5$. The following expressions are, of course, not at all surprising:

$$H^*(M_1, \mathbb{Q}) = \mathbb{Q}[\omega_2] / (\omega_2^6) \quad (2.40)$$

$$H^*(M_2, \mathbb{Q}) = \mathbb{Q}[\omega_1] / (\omega_1^6). \quad (2.41)$$

3b) $m_1 = m_2 = 1$.

From the results obtained at 3a) we get immediately:

$$H^*(M, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \omega_2, \omega_{212}]/(\omega_1^2 + \omega_2^2 + \omega_1\omega_2, \omega_1^3, \omega_2^3, \omega_{212}^2) \quad (2.42)$$

$$H^*(M_1, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_2, \omega_{212}]/(\omega_2^3, \omega_{212}^2) \quad (2.43)$$

$$H^*(M_2, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \omega_{121}]/(\omega_1^3, \omega_{121}^2), \quad (2.44)$$

where $\deg \omega_1 = \deg \omega_2 = 1$, and $\deg \omega_{212} = \deg \omega_{121} = 3$.

2.5 Uniform multiplicity 1 or 2

Two types of homogeneous spaces will be considered here: principal adjoint orbits of connected compact simple Lie groups and principal isotropy orbits of the so-called symmetric spaces of maximal rank (i.e., in our usual notations, irreducible symmetric spaces of non-compact type of the form G/K for which a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is also maximal abelian in $\mathfrak{g}$). The first ones constitute the class of homogeneous isoparametric submanifolds of uniform multiplicity 2, so we will be interested about their cohomology with coefficients both in $\mathbb{Z}$ and $\mathbb{Z}_2$; it is also clear that the other ones have uniform multiplicity 1 (they are actually the only isoparametric submanifolds with this property), so all could we hope is only to describe their cohomology with coefficients in $\mathbb{Z}_2$. The reason because they are considered together is that they are the same from the viewpoint of the second coloured Dynkin diagram and implicitly of their $\mathbb{Z}_2$ -cohomology rings (cf. Thm.2.3.12, d)). Explicitely, we will use the following statement, which follows trivially from the theorem cited above:

Proposition 2.5.1 *Let G be a compact connected simple Lie group which has the same root system as the symmetric space of maximal rank G'/K . Then the $\mathbb{Z}_2$ -cohomology ring of a principal isotropy orbit of G'/K can be obtained from the $\mathbb{Z}_2$ -cohomology ring of a principal adjoint orbit of G after dividing the dimensions of generators by 2.*

Even if it has no relevance for our present developement, we mention that in the assertion above the symmetric space could has been supposed of compact type (cf. Remark 1., section 1.2) and in this case, we could have supposed that $G = G'$: one can associate to any simple, compact, connected

and simply connected Lie group G in a standard manner an involutive automorphism of fixpoint set K , such that G/K is a symmetric space of maximal rank; any compact symmetric space of maximal rank can be obtained in this way (see [25]).

As about the principal adjoint orbits, they are quotients of the form G/T , where T is a maximal torus in G , and it is maybe useful to follow briefly the progresses that have been made during the time in investigating the cohomology of this spaces. For the rational cohomology, a very interesting formula has been obtained by A. Borel in [3], namely:

$$H^*(G/T, \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_k] / \langle \mathbb{Q}^+[x_1, \dots, x_k] \rangle \quad (2.45)$$

where $k = \dim T$ is the rank of G , W its Weyl group and $\langle \mathbb{Q}^+[x_1, \dots, x_k] \rangle$ the ideal generated by the W -invariant homogeneous polynomials of positive degree in $\mathbb{Q}[x_1, \dots, x_k]$. Also to A. Borel is due the remark that a similar formula holds if $\mathbb{Q}$ is replaced by $\mathbb{Z}$, but only under the assumption that $H^*(G)$ has no torsion, G being supposed simply connected (see Prop.29.2 of the paper cited above). But in order to have no torsion, G must be of A_k or C_k type, i.e. $SU(k+1)$ or $Sp(k)$. As we already mentioned in our previous section, a formula for $H^*(G/T, \mathbb{Z})$ when G is of G_2 type was obtained by Bott and Samelson in [5]. The types B_k , D_k , F_4 and E_6 were elucidated by Toda and Watanabe in [41], also from the point of view of the integral cohomology of G/T . As about the remaining cases when G is of type E_7 or E_8 , Kono and Ishitoya obtained in [23] formulae which describe the cohomology of G/T , but only when the coefficient ring is $\mathbb{Z}_2$, and probably precise descriptions of their integral cohomology are not known so far.

We will consider in the sequel one by one all "monocolor" second coloured Dynkin diagrams with at least three vertices, denote for each of them by M_1 and M_2 the corresponding isoparametric submanifolds of uniform multiplicity 1, respectively 2, and, whenever possible, give descriptions of $H^*(M_1, \mathbb{Z}_2)$ and $H^*(M_2, \mathbb{Z})$.

1. The A_k type.

Taking into account that $SU(k+1)$ has no torsion, it follows easily from the already mentioned Prop.29.2 of [3] and some considerations regarding the action of the Weyl group on the set of homogeneous polynomials, that:

$$H^*(M_2, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_{k+1}] / \langle \sigma_1(x), \sigma_2(x), \dots, \sigma_{k+1}(x) \rangle, \quad (2.46)$$

where $\sigma_j(x)$ denotes the j -th elementary symmetric polynomial in the variables $x_1, \dots, x_{k+1}$ and $\deg x_j = 2$, $1 \leq j \leq k+1$ (see for instance [27], Ch.III, Thm.5.5). Consequently,

$$H^*(M_1, \mathbb{Z}_2) = \mathbb{Z}_2[y_1, \dots, y_{k+1}] / \langle \sigma_1(y), \sigma_2(y), \dots, \sigma_{k+1}(y) \rangle, \quad (2.47)$$

where y_j is this time of degree 1, $1 \leq j \leq k+1$.

2. The C_k type.

Also $Sp(k)$ has no torsion, hence, in the same manner as before, we obtain:

$$H^*(M_2, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_k] / \langle \bar{\sigma}_1(x), \bar{\sigma}_2(x), \dots, \bar{\sigma}_k(x) \rangle, \quad (2.48)$$

where $\bar{\sigma}_j(x)$ means $\sigma_j(x_1^2, \dots, x_k^2)$, $\deg x_j = 2$, $1 \leq j \leq k$ and

$$H^*(M_1, \mathbb{Z}_2) = \mathbb{Z}_2[y_1, \dots, y_k] / \langle \bar{\sigma}_1(y), \bar{\sigma}_2(y), \dots, \bar{\sigma}_k(y) \rangle, \quad (2.49)$$

with $\deg y_j = 1$, $1 \leq j \leq k$.

3. The B_k and D_k types with uniform multiplicity 2.

The following two descriptions are obtained in [41]:

$$H^*(SO(2k+1)/T, \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_k, e_2, \dots, e_{2k}] / I_k, \quad (2.50)$$

I_k being the ideal generated by

$$\{\sigma_i - 2e_{2i}, e_{4i} + \sum_{0 < j < 2i} (-1)^j e_{2j} e_{4i-2j} : i = 1, 2, \dots, k\},$$

where $\deg t_i = 2$, $\deg e_{2i} = 2i$, $e_{2j} = 0$, for $j > k$. As about the D_k type, the following description can be founded in the same paper:

$$H^*(SO(2k)/T, \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_k, e_2, \dots, e_{2k-2}] / J_k, \quad (2.51)$$

J_k the ideal generated by

$$\{\sigma_i - 2e_{2i}, e_{4i} + \sum_{0 < j < 2i} (-1)^j e_{2j} e_{4i-2j} : i = 1, 2, \dots, k-1\} \cup \{\sigma_k\}$$

where $\deg t_i = 2$, $\deg e_{2i} = 2i$, $e_{2j} = 0$, for $j \geq k$. As usual, $\sigma_i = \sigma_i(t_1, \dots, t_k)$ denotes the i -th elementary symmetric polynomial in $t_1, \dots, t_k$, $1 \leq i \leq k$.

4. The B_k and D_k types with uniform multiplicity 1.

All we need is the $\mathbb{Z}_2$ -cohomology of $SO(2k+1)/T$ and $SO(2k)/T$. We could, of course, tensorize with $\mathbb{Z}_2$ their integral cohomology rings, known by the formulas above. By reasons that will become clear only later, when we will try to calculate the so-called cup-length of the cohomology rings obtained in this section, we prefer to choose another method, in order to get formulas which are more convenient with this respect.

Namely we begin by the description of the $\mathbb{Z}_2$ -cohomology of the symmetric space $SO(2k)/U(k)$ given in [27], Ch.III, Th.6.11: this algebra is simply generated by certain elements $e_2, \dots, e_{2k-2}$, subjects of the only relations:

$$e_{4j} + \sum_{0 < i < j} e_{2i} e_{4j-2i} = 0,$$

for all $1 \leq j \leq k-1$, where e_{2j} must be readed as 0, whenever $j \geq k$. Then we notice that for both of the following two bundles:

$$\begin{aligned} U(K)/T &\rightarrow SO(2k)/T \rightarrow SO(2k)/U(k), \\ SO(2k)/U(k) &\rightarrow SO(2k+1)/U(k) \rightarrow S^{2k} \end{aligned}$$

the base-space and the fiber have vanishing $\mathbb{Z}_2$ -cohomology in odd dimensions. It follows by a trivial argument of spectral sequences that the cohomology of the total space is in both cases the tensor product of the cohomology algebras of the base-space and the fiber, i.e.:

$$H^*(SO(2k)/T, \mathbb{Z}_2) = H^*(U(k)/T, \mathbb{Z}_2) \otimes H^*(SO(2k)/U(k), \mathbb{Z}_2), \quad (2.52)$$

$$H^*(SO(2k+1)/T, \mathbb{Z}_2) = H^*(SO(2k)/T, \mathbb{Z}_2) \otimes H^*(S^{2k}, \mathbb{Z}_2). \quad (2.53)$$

All factors of the products from the right-hand sides above are already known. If the Weyl group of M_1 is of B_k , respectively D_k type, then $H^*(M_1)$ can be obtained from (2.49), respectively (2.48) after dividing the dimensions of the generators by 2.

5. The F_4 and E_6 types.

Precise formulas for $H^*(M_2, \mathbb{Z})$ are given in [41], Thms.A and B. It is also clear how can be derived from here $H^*(M_1, \mathbb{Z}_2)$.

6. The E_7 and E_8 types.

The $\mathbb{Z}_2$ -cohomology of M_2 is described in [23], Thm.3.10; divide by 2 the dimensions of the generators in order to obtain $H^*(M_1, \mathbb{Z}_2)$.

Remark Although it doesn't belong strictly to the context of the present section, consider here also the situation when M is of A_k type with uniform multiplicity 4: by Cor.2.3.10, we obtain a description of $H^*(M, \mathbb{Z})$ if we shift in (2.42) the common value of the degrees of x_j , $1 \leq j \leq k+1$, from 2 to 4.

2.6 The Weyl group of B_k type and different multiplicities

The goal of the present section is to apply what we like to call the "prototype principle" (more precisely, Cor. 2.3.8) in order to calculate the $\mathcal{R}$ -cohomology ring of all homogeneous isoparametric submanifolds whose first Dynkin diagram has one of the types $B_k III = C_k III$ or $(BC)_k IV = C_k IV$. As an immediate consequence, the $\mathbb{Z}_2$ cohomology of all isoparametric submanifolds of second Dynkin diagram $B_k II = C_k II = (BC)_k II$ will be obtained. Which are exactly the symmetric spaces whose principal isotropy orbits are involved here, it can be easily seen by consulting the three tables from the Appendix together with Table VI and Table V from [18]. It will be enough to find among this spaces a satisfactory one, if possible one whose isotropy orbit is accessible from the viewpoint of the $\mathbb{Z}$ -cohomology. As we will see in the sequel, a such representative is furnished by the symmetric space $SU(2k+m)/S(U(k) \times U(k+m))$.

We will not get into the details concerning this symmetric space: they can be founded for instance in [18], Ch.X, § 2, §§ 3. Mention only that it is a symmetric space of $(BC)_k$ type, whose marked Dynkin diagram is:

$$(BC)_k \quad \overset{2}{\circ} \text{---} \overset{2}{\circ} \dots \overset{2}{\circ} \text{---} \overset{2}{\circ} \text{---} \overset{2m+1}{\circ}$$

and the concrete form of a principal isotropy orbit is :

$$M_0 = \frac{S(U(k) \times U(k+m))}{S(\Delta(S^1 \times \dots \times S^1) \times U(m))} \quad (2.54)$$

where $\Delta(S^1 \times \dots \times S^1) = \{\text{Diag}(z_1, \dots, z_k, z_1, \dots, z_k) : |z_1| = \dots = |z_k| = 1\}$
We simplify the expression of M_0 in two steps. First we apply the following

general result:

Lemma 2.6.1 *Associate to the closed connected subgroups H and L of $U(m)$ the subgroups*

$$SH = \{x \in H : \det x = 1\}$$

and

$$SL = \{x \in L : \det x = 1\}.$$

If any matrix of the form $\text{Diag}(z, 1, \dots, 1)$, $z \in \mathbb{C}$, $|z| = 1$, is in H and H is a subgroup of L , then the homogeneous spaces L/H and SL/SH are diffeomorphic.

PROOF A such diffeomorphism is given by:

$$L/H \rightarrow SL/SH, [x] \mapsto [x \text{Diag}(\frac{1}{\det x}, 1, \dots, 1)], x \in L.$$

□

Consequently,

$$M_0 \simeq \frac{U(k) \times U(k+m)}{\Delta(S^1 \times \dots \times S^1) \times U(m)}.$$

Then we notice that the correspondance $[(g, h)] \mapsto ([g], [h(g^{-1}, 1)])$, $(g, h) \in U(k) \times U(k+m)$ gives the diffeomorphism

$$\frac{U(k) \times U(k+m)}{\Delta(S^1 \times \dots \times S^1) \times U(m)} \simeq \frac{U(k)}{S^1 \times \dots \times S^1} \times \frac{U(k+m)}{U(m)} = \frac{U(k)}{T} \times W_{m+k,k},$$

where $T = S^1 \times \dots \times S^1 = \{\text{Diag}(z_1, \dots, z_k) : |z_1| = \dots = |z_k| = 1\}$ is a maximal torus in $U(k)$ and $W_{m+k,k} = U(k+m)/U(m)$ the complex Stiefel manifold consisting of systems of k orthonormal vectors in $\mathbb{C}^{k+m}$.

Both $W_{m+k,k}$ and $U(k)/T$ have no torsion and moreover, by Prop.9.1 of [3], we have:

$$H^*(W_{m+k,k}, \mathbb{Z}) = H^*(S^{2(m+k)-1} \times S^{2(m+k)-3} \times \dots \times S^{2m+1}, \mathbb{Z}) \quad (2.55)$$

If in our previous formula (2.42) we replace $k+1$ by k , we obtain the integral cohomology of $U(k)/T$. Conclude by:

$$H^*(M_0, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_k] / (\sigma_1(x), \dots, \sigma_k(x)) \otimes \Lambda(e_1, \dots, e_k),$$

with $\deg x_j = 2$ and $\deg e_j = 2m + 2j - 1$, for any $1 \leq j \leq k$.

Let now M be an arbitrary isoparametric submanifold of rank k at least 3 whose marked Coxeter graph is of the form:

$$\begin{array}{ccccccc} m_1 & m_1 & m_1 & m_1 & m_2 \\ \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$$

with $m_1 \neq m_2$. Consider first the case when m_1 and m_2 are both greater than 1. By a direct observation of Table 1. from the Appendix it follows that m_1 is even, m_2 is odd and they constitute relatively prime integers. We conclude from Cor.2.3.7 that:

$$H^*(M, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_k] / (\sigma_1(x), \dots, \sigma_k(x)) \otimes \Lambda(e_1, \dots, e_k), \quad (2.56)$$

with $\deg x_j = m_1$ and $\deg e_j = m_2 + (k - j)m_1$, for any $1 \leq j \leq k$.

Let us allowd now to m_1 or m_2 to be equal to 1 and estimate the $\mathbb{Z}_2$ -cohomology of M . By Thm.2.3.11, d), follows:

$$H^*(M, \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \dots, x_k] / (\sigma_1(x), \dots, \sigma_k(x)) \otimes \Lambda(e_1, \dots, e_k), \quad (2.57)$$

where once again $\deg x_j = m_1$ and $\deg e_j = m_2 + (k - j)m_1$, for any $1 \leq j \leq k$ (of course, Λ denotes in (2.53) the exterior algebra over $\mathbb{Z}$ and in (2.54) the exterior algebra over $\mathbb{Z}_2$).

2.7 The Weyl group of F_4 type and different multiplicities

In order to have a complete image about the $\mathbb{Z}_2$ -cohomology of irreducible isoparametric submanifolds, the only missing piece of information regards the situation when the second Dynkin diagram is of F_4II type. More precisely, there are three such principal isotropy orbits: the corresponding symmetric spaces, together with brief descriptions of the orbits we are talking about are given in [19], section 3. By our Thm.2.3.12, it would be enough to obtain the $\mathbb{Z}_2$ -cohomology for any of these orbits, and then to apply the same scheme as in the previous section. Unfortunately, none of them seems us accessible enough with this respect. First of all it is not easy to understand

the exact structure as quotient spaces of these isotropy orbits, even with the indications given in the paper cited above. Supposing that this task would be finally done, we can expect to obtain quotients whose cohomology is difficult to be described.

All could we done in this direction was to describe completely the principal isotropy orbits of one of these three symmetric space, but without succeeding in calculating its cohomology. Namely, the symmetric space is $E_6/SU(6) \times SU(2)$. A certain amount of calculations is needed in order to obtain a principal stabilizer in the corresponding isotropy representation of $SU(6) \times SU(2)$: Denote by T^2 the standard maximal torus in $SU(3)$, namely the set of all matrices of the form $\text{Diag}(z_1, z_2, z_3)$ with $|z_1| = |z_2| = |z_3| = 1$ and $z_1 z_2 z_3 = 1$. A principal stabilizer is then $H = H_0 \sqcup (J, j)H_0 \subseteq SU(6) \times SU(2)$, where the connected component of the unit H_0 is:

$$H_0 = \left\{ \begin{pmatrix} \eta t & 0 \\ 0 & \bar{\eta} t \end{pmatrix} : t \in T^2, \eta \in \mathbb{C}, \eta^4 = 1 \right\}$$

and

$$J = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix}, j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Chapter 3

Lusternik-Schnirelmann category of isoparametric submanifolds

After evaluating their homology module and cohomology algebra, we would like now to investigate the isoparametric submanifolds from the point of view of other topological or differential invariants, namely the Lusternik-Schnirelmann category and the functional category. Among the isoparametric families, we distinguish the class of adjoint orbits and for these ones we will be able to calculate both of these invariants.

3.1 Basic results and primar estimations

In a very brief manner we summarize in the sequel some classical facts of the Lusternik-Schnirelmann theory, together with some related results of Takens' paper [37] concerning the so-called functional category.

Recall that the *Lusternik-Schnirelmann category* $\text{cat}(X)$ (or simply, the *L.S.-category*) of a topological space X is the minimal cardinality k of an open covering $\{X_1, \dots, X_k\}$ of X such that each X_j is contractible in X ; if it happens that a such covering does not exist, we take $\text{cat}(X)=\infty$. It is a homotopical invariant, difficult to calculate for concrete examples (although the two general results which will follow in the sequel can be often helpful in furnishing lower, respectively upper bounds). Closely related is also the notion of cup-length: If $\mathcal{R}$ is an arbitrary ring, we call the $\mathcal{R}$ -cup-length (or

the $\mathcal{R}$ -cohomological length) of the space X and denote by $\text{cuplength}_{\mathcal{R}}(X)$ the maximal number of elements of $H^*(X, \mathcal{R})$ of degree at least 1 whose cup-product is nonzero.

Theorem 3.1.1 *For any space X and any ring $\mathcal{R}$ holds:*

$$1 + \text{cuplength}_{\mathcal{R}}(X) \leq \text{cat}(X).$$

Theorem 3.1.2 *Let X be a finite CW-complex. If X is $r - 1$ -connected, where r is an integer greater than 1, then:*

$$\text{cat}(X) \leq 1 + \frac{1}{r} \dim X.$$

Proofs of these theorems and other facts concerning the notion of L.S.-category can be founded for instance in the monography [21] of I.M. James.

Let us restrict now to the case of finite dimensional, closed manifolds. If M is a such one, by following [37] we denote by $F(M)$ and call the F -category of M the minimal number of critical points that a real function on M can have. Relevant for the difficulties encountered in calculating $F(M)$ is the fact that the statement " $\text{cat}(M^n)=2$ imply $F(M^n)=2$ " is equivalent to the conjecture of Poincarè (without getting into details, we only notice that $\text{cat}(M^n)=2$ iff M^n is a homotopy n -sphere and $F(M^n)=2$ iff M^n is homeomorphic to a n -sphere).

The following statement goes back to the original work of Lusternik and Schnirelmann from 1934; it can also be interpreted as a topological obstruction for the differentiable structure that M could carry.

Theorem 3.1.3 *For any closed manifold M holds:*

$$\text{cat}(M) \leq F(M).$$

It is interesting to notice that there exist so far no examples of closed manifolds for which the inequality above is strictly. On the other hand, it seems difficult to prove that the equality holds in general (some progresses in solving this problem have been recently made by O. Cornea in [9]). All could be done for the moment is probably to analyze from this perspective particular classes of closed manifolds: one of the aims of this chapter is to do that for the class of adjoint orbits of compact connected semisimple Lie groups. An important tool we will use to this end is the theorem stated

below, which represents one of the main results of [37]. Consider first the following list of properties that a set E can satisfy relatively to M :

- a) $E \subset \mathbb{Z}$;
- b) $n - 3, n - 2, n - 1 \in E$;
- c) if $H_i(M) \neq 0$ then $i \in E$;
- d) if $H_i(M)$ has torsion, then $i + 1 \in E$.

Theorem 3.1.4 *Let M be an n -dimensional, p -connected closed manifold with $n \geq 6$ and $p \geq 1$. If there exists a set E which satisfies the conditions a)-d) of above, E contained in the union of s closed intervals of length p , then $F.(M) \leq s$.*

We will try in the remaining part of the section to apply the results exposed above to the particular context of isoparametric submanifolds and to see how far can we go in investigating them when using only such general facts. The first thing that can be done is, of course, to use the results obtained during the sections 2.4, 2.5 and 2.6: they reduce the problem of calculating $\text{cuplength}_{\mathbb{Z}_2}(M)$, M irreducible isoparametric submanifold, to an algebraic case-by-case computation.

Let us begin with the case when M is of rank 2: even the value of $\text{cuplength}_{\mathcal{R}}(M)$ can be then obtained, with $\mathcal{R}$ chosen in the usual manner in terms of the values of multiplicities (mention that, at least theoretically, for spaces without torsion, the $\mathbb{Z}$ -cuplength approximates L.S.-cat better than $\mathbb{Z}_2$ -cuplength). We arrange the informations obtained in this way in the following table:

Type	Multiplicities	$\mathcal{R}$	Cuplength $_{\mathcal{R}}$
A_2	$m_1 = m_2 \geq 2$	$\mathbb{Z}$	3
	$m_1 = m_2 = 1$	$\mathbb{Z}_2$	3
B_2	$m_1 = m_2 = 2$	$\mathbb{Z}$	4
	$m_1 = m_2 = 1$	$\mathbb{Z}_2$	4
	$1 < m_1 < m_2$	$\mathbb{Z}$	3
	$1 = m_1 < m_2$	$\mathbb{Z}_2$	3
G_2	$m_1 = m_2 = 2$	$\mathbb{Z}$	6
	$m_1 = m_2 = 1$	$\mathbb{Z}_2$	4

We will come back with some more facts about the L.S.-category of these spaces in the last section of the chapter.

The next case that we consider is that when the multiplicities are equal and the rank greater than 2. As we already explained in section 2.5, we can uniquely associate to any irreducible root system two symmetric spaces whose principal isotropy orbits M_1 , respectively M_2 , have uniform multiplicity 1, respectively 2. We can easily see that if M_2 is the principal adjoint orbit of the Lie group G , then $\text{cuplength}_{\mathbb{Z}}(M_2)$ equals the number of positive roots of G . The obtention of the common value of $\text{cuplength}_{\mathbb{Z}_2}(M_1)$ and $\text{cuplength}_{\mathbb{Z}_2}(M_2)$ is generally not any more so simple, although reduced to a certain amount of algebraic calculations. The most reasonable cases are A_k and C_k , since then $\text{cuplength}_{\mathbb{Z}_2}(M_2)$ and $\text{cuplength}_{\mathbb{Z}}(M_2)$ are the same. The latter identity fails to be true in general, as can be seen for instance in the D_k and B_k cases: from (2.52), respectively (2.53) it follows that $\text{cuplength}_{\mathbb{Z}_2}(M_2)$ equals $1/2(k-1)(k+2)$, respectively $1/2(k-1)(k+2) + 1$, which is smaller than $k(k-1)$, respectively k^2 , provided $k \geq 3$. In the remaining situation when the common value of multiplicities is 4, the isoparametric submanifold M must be of A_k type; associate the corresponding M_2 and notice that by Cor.2.3.10, $\text{cuplength}_{\mathbb{Z}}(M) = \text{cuplength}_{\mathbb{Z}}(M_2)$.

Finally, when the Weyl group of M is of B_k type and the two multiplicities are different, we apply (2.57) and deduce immediately that $\text{cuplength}_{\mathbb{Z}_2}(M)$ equals $1/2k(k+1)$. We recall that none of the three isoparametric submanifolds with Weyl group of F_4 type and different multiplicities was accessible from the point of view of the $\mathbb{Z}_2$ -cohomology, hence we see no way to calculate the (common) value of their $\mathbb{Z}_2$ -cohomological length.

Of course, the estimations just made can offer only a lower bound for the L.S.-category of the spaces involved in our discussion. The obtention of $\text{cat}(M)$, where M is an arbitrary irreducible isoparametric submanifold, seems to be in general a difficult problem and we cannot expect to obtain too much in searching this value with the minimal background of L.S.-theory that we only dispose for the moment. But something still can be done, namely we can, in a quite elementary manner, determine the L.S.-category for adjoint orbits of compact Lie groups and that, not only for principal orbits of simple groups, but, in general, for arbitrary adjoint orbits of connected semisimple Lie groups. Singhof pointed out in [35] that if G is a compact connected semisimple Lie group and $T \subseteq G$ a maximal torus then on the one hand, $H^*(G/T, \mathbb{Q})$ is generated as algebra by 1 and its elements of degree 2 and on the other hand, G/T is simply connected. Hence Thms.3.1.1 and 3.1.2 can be invoked in order to obtain:

$$\text{cat}(G/T) = 1 + 1/2\dim(G/T) = 1 + \text{Card}(\Delta^+),$$

where Δ^+ denotes a system of positive roots of G . Of course, we cannot hope that any adjoint orbit of the form $\text{Ad}(G).x$, with x in the Lie algebra of G , has the cohomology algebra generated by its elements of degree 2 (mention only that the Grassmannian $G_n(\mathbb{C}^N)$ of complex n -planes in $\mathbb{C}^N$ is a such orbit). So the idea to reproduce *ad literam* the argument above doesn't work. Another interesting particularity of the adjoint orbits of G is that they are all Kähler manifolds (see for instance [4], Thm.2), hence on each of them there exists a closed 2-form whose appropriate power is the volume form. The $\mathbb{R}$ -cuplength of $\text{Ad}(G).x$ must be in this way at least half of its dimension. Since these orbits are also simply connected, we can state:

$$\text{cat}\text{Ad}(G).x = 1 + 1/2\dim(\text{Ad}(G).x).$$

The previous reasoning can be summarized by saying that, for the L.S.-category of adjoint orbits, the lower bound given by Thm.3.1.1 and the upper bound given by Thm.3.1.2 coincide. By Thm. 3.1.4, the same value constitutes also an upper bound for the F-category (hence its exact value), provided that the dimension of the orbit is at least 6. In fact, if $n = 2m$ denotes the dimension of $\text{Ad}(G).x$, we take into account that In fact, we must only notice that the arbitrary adjoint orbit $\text{Ad}(G).x$ has no torsion and vanishing integral homology in all odd dimensions. Hence, if $n = 2m$ denotes the dimension of the orbit, then the set $E = \{0, 2, 4, \dots, n-4, n-3, n-2, n-1, n\}$ satisfies the properties a)-d) above. But E can be covered in a trivial manner by $m + 1$ closed intervals of length 1. Consequently:

Proposition 3.1.5 *Let $\text{Ad}(G).x$ be an arbitrary adjoint orbit of the compact connected semisimple Lie group G . Then:*

$$1 + \text{cuplength}_{\mathbb{Z}}\text{Ad}(G).x = \text{cat}\text{Ad}(G).x = 1 + 1/2\dim\text{Ad}(G).x.$$

Moreover, if the dimension of the orbit is at least 6, then also its F.-category equals the same value.

The question is now which is the F.-category for adjoint orbits of dimension 2 or 4. The goal of the next section is to prove that the equality between F-category and L.S.-category still remains valid for these orbits.

3.2 Adjoint orbits of dimension less than 6 and their F.-category

The following two technical results will be needed to obtain restrictions on the structure of adjoint orbits of dimension 2, respectively 4. All root systems we are referring here are supposed to be reduced.

Proposition 3.2.1 *Let R be a root system of rank at least 2 in the Euclidean space V . Let $R = \bigcup R_\nu$, $V = \bigoplus V_\nu$ be the splitting of (R, V) as a direct sum of irreducible root systems. Let R^+ be the set of positive roots and let $\Gamma_1 \subset R^+$ be the complement in R^+ of a certain positive root α . If $\text{span}\Gamma_1 \neq V$, then α belongs to a direct summand of type A_1 .*

Proposition 3.2.2 *Let Γ_2 be the complement in R^+ of the two different positive roots α and β in the same root system (R, V) as before. Suppose that $\text{span}\Gamma_2 \neq V$. Then one of the following statements is true:*

- a) both α and β belong to the same summand, which is of type A_2 ;
- b) α or β belongs to a summand of type A_1 .

Both propositions are direct consequences of :

Lemma 3.2.3 *Let R be an irreducible root system of rank at least 3 in the Euclidean space V and let m be the number of positive roots. Then the subspace spanned by any $m - 2$ positive roots is the whole V .*

PROOF The positive roots we are referred are related to a certain system of simple roots, say it $S = \{\gamma_1, \dots, \gamma_k\}$. Take α, β two positive roots and consider $\Gamma = R^+ \setminus \{\alpha, \beta\}$. Suppose that the span of Γ is $V_\Gamma \neq V$. Distinguish the following two situations:

- (i) α and β are both simple. Say $\alpha = \gamma_{k-1}$ and $\beta = \gamma_k$.

For any $1 \leq i \leq k - 2$, we have $\langle \gamma_i, \alpha \rangle = 0$: Otherwise $\gamma_i + \alpha \in R^+$, hence $\gamma_i + \alpha \in \Gamma$ and that means $\alpha \in V_\Gamma$. It follows that for any $1 \leq j \leq k - 2$, $\langle \gamma_j, \beta \rangle = 0$ (otherwise $\beta \in V_\Gamma$ and further $V_\Gamma = V$). If $\langle \beta, \alpha \rangle = 0$, then β is orthogonal to $S \setminus \{\beta\}$, which contradicts the irreducibility of R . Since $\langle \beta, \alpha \rangle \neq 0$, $\beta + \alpha \in \Gamma$ and in this way, $\beta \in V_\Gamma$; but we have already seen that this is impossible.

Similarly, $\langle \gamma_i, \beta \rangle = 0$, for any $1 \leq i \leq k - 2$. But this contradicts clearly the irreducibility of R .

(ii) among α and β , only one is simple. Say $\beta = \gamma_k$.

Clearly $\beta \notin V_\Gamma$ (otherwise V_Γ would contain all simple roots, being equal to V).

There exists a certain $i \in \{1, \dots, k-1\}$, with $\langle \beta, \gamma_i \rangle \neq 0$: for this one, holds also $\beta + \gamma_i \in R^+ \setminus \Gamma$, so $\beta + \gamma_i = \alpha$. In other words, there exists in $S \setminus \{\beta\}$ a unique root, say it γ_{k-1} , with $\langle \beta, \gamma_{k-1} \rangle \neq 0$; it must also satisfy $\alpha = \beta + \gamma_{k-1}$. Notice that $\alpha \notin V_\Gamma$: otherwise, from $\gamma_{k-1} \in V_\Gamma$, would follow $\beta = \gamma_k \in V_\Gamma$.

Let's prove that for any $1 \leq i \leq k-2$, $\langle \alpha, \gamma_i \rangle = 0$. Suppose that $\langle \alpha, \gamma_i \rangle \neq 0$, for a certain $1 \leq i \leq k-2$. It would follow that $\alpha + \gamma_i$ must be in R^+ , so in Γ and implicitly in V_Γ , hence $\alpha \in V_\Gamma$, contradiction.

The conclusion is the same as in the previous case, namely that for any $1 \leq i \leq k-2$, $\langle \gamma_i, \gamma_{k-1} \rangle = \langle \gamma_i, \gamma_k \rangle = 0$, and this is, of course, impossible. □

We are now in position to give the expected estimations (the Lie groups whose orbits are involved in the following enouncements are supposed, as usual, compact, connected and semisimple). The main question of the section has a trivial answer if the dimension of the orbit is 2:

Proposition 3.2.4 *Any 2-dimensional adjoint orbit of a compact connected Lie group is diffeomorphic to the sphere S^2 ; consequently, its category equals 2.*

As about the four dimensional situation, we will prove:

Proposition 3.2.5 *On any four dimensional isotropy orbit of a compact connected semisimple Lie group there exists a function with 3 critical points.*

PROOF Let G be a compact connected semisimple Lie group, $T \subseteq G$ a maximal torus and $\mathfrak{t} \subseteq \mathfrak{g}$ their Lie algebras. Equipped with the restriction of the negative of Killing form, $\mathfrak{t}$ becomes Euclidean space: the set Δ of all real roots is a reduced root system in $\mathfrak{t}^*$. The duals of the elements of Δ , constitute the set Δ^* , which is a root system in $\mathfrak{t}$. If $x \in \mathfrak{t}$, then the dimension of $\text{Ad}(G).x$ equals the number of positive roots which do not vanish x . Hence $\dim(\text{Ad}(G).x)=4$ means that there are in Δ^* exactly two positive roots, say α^* , β^* , which are not orthogonal to x . By Prop. 3.2.2, only the following two situations are possible:

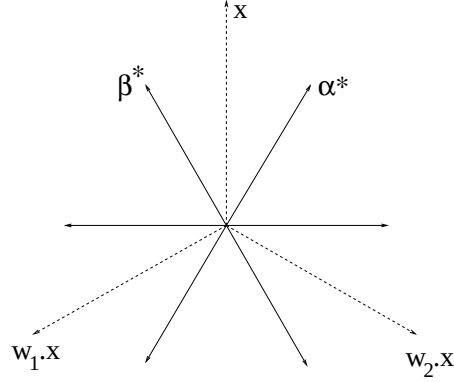


Figure 3.1.

a) α^* and β^* are both in a direct summand (V_0, R_0) of type A_2 . Also x must be in V_0 , orthogonal to the third positive root from R_0 (namely those different of α^* and β^*).

If a is a regular point in $\mathfrak{t}$, then the distance function $f_a : \text{Ad}G.x \rightarrow \mathbb{R}$ has as critical set the orbit $W.x$, where W is the Weyl group of G . The number of these critical points is clearly 3 (also see Figure 3.1).

b) α^* and β^* is each in a direct summand of type A_1 and $x \in \text{span}\{\alpha^*, \beta^*\}$ (we can suppose, by Prop.3.2.2, that α^* is in a direct summand (V_1, R_1) of type A_1 ; if (V_2, R_2) is the summand which contains β^* , then the orthogonal projection of x onto V_2 is a vector from V_2 orthogonal to all positive roots of R_2 , except β^* , which means, by Prop.3.2.1, that also (V_2, R_2) is of type A_1).

This time x is a regular point, hence G has only two positive roots, i.e. it is of type $A_1 \times A_1$. The adjoint orbit is evidently the product of the corresponding two principal adjoint orbits, so just a product of two 2-dimensional spheres.

One can obtain a function on $S^2 \times S^2$ which has only 3 critical points, by the same method which Takens used in [37], Statement 1, p.209, in order to construct a function with the same property on $S^1 \times S^2$ (without getting into details, we only mention that the construction can be, in fact, easily extended to any product of two positive-dimensional spheres).

□

We can now conclude by:

Corollary 3.2.6 *Let G be a compact connected semisimple Lie group and*

$\text{Ad}(G).x$ an arbitrary adjoint orbit. Then:

$$F(\text{Ad}(G).x) = \text{cat}(\text{Ad}(G).x) = 1 + 1/2\dim(\text{Ad}(G).x).$$

3.3 Open questions

We come back to the general question of finding out the L.S.-category for all isoparametric submanifolds. As we have already seen, a lower bound given by the $\mathbb{Z}_2$ -cohomological length can be, at least theoretically, obtained. The particular situation when multiplicities are equal and even is a fortunate one, the multitude of particularities present here furnish even the value of F.-category. We cannot expect to obtain similar results for the other cases, at least not by using the same methods.

It would be interesting to know the L.S.-category of the isoparametric submanifolds of type B_2 (i.e isoparametric hypersurfaces in the sphere with 4 distinct principal curvatures). This is the only situation where nonhomogeneous examples are known, namely these constructed by Ferus, Karcher and Münzner in [13]. Moreover, it is conjectured that the only isoparametric submanifolds of this type are those who occur in the paper mentioned above. The study of this case from various points of view, can be, of course, useful. Actually, the case is classified from the point of view of the so-called rational homotopy type. It makes no sense to get now into the details of the rational homotopy theory: we just mention that to any simply connected space X we can associate, together with the usual L.S.-category, the so-called it rational L.S.-category, denoted by $\text{cat}_0(X)$, which depends only on the rational homotopy type of X and is, in general, smaller or equal to $\text{cat}(X)$ (for more details, can be seen for instance [2], Ch.V). In [15] the authors obtained the rational hotopy type for all rank 2 isoparametric submanifolds: in the case B_2 , for instance, when the multiplicities m_1 and m_2 are different and greater than 2, the corresponding manifold M has the rational homotopy type of the product $S^{m_1} \times S^{m_2} \times S^{m_1+m_2}$ (this means simply that the homotopy groups of these two spaces are the same, up to a tensor product with $\mathbb{Q}$). It follows $\text{cat}_0(M) = 4$, hence $\text{cat}(M) \geq 4$ (the same lower bound could have been obtained by using the $\mathbb{Z}_2$ -cuplength, see the table from section 3.1).

On the other hand, we can obtain on the same M a function with only 5 critical points, in the following way: At an arbitrary point x of M , we consider the normal space $\nu_x(M)$, together with the four focal hyperplanes.

Without loss of generality we can suppose that x is at equal distances of the walls $l_1(x)$ and $l_2(x)$ which bounds its Weyl chamber (we can eventually map M diffeomorphically, by an easy to imagine end-point map, onto a manifold which satisfies this request). The distance function from x restricted on m has as critical set the W -orbit of x (see Figure 3.2 below). Notice that the two points of index 1, 2, and 3, are respectively on same levels.

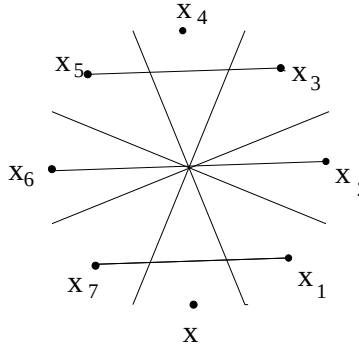


Figure 3.2

By the convexity theorem of Terng (see for instance [32], Thm.8.6.4), the orthogonal projection of M on the affine 2-plane $\nu_x(M)$ is the octagon of vertices $x, x_1, \dots, x_7$. Consequently, we can find a smooth curve in M , whose orthogonal projection is the segment $[x_1, x_7]$, and is consequently contained in the same level of f as x_1 and x_7 . The same can be said about the pairs of points x_2, x_6 and x_3, x_5 . We apply now the result stated by Takens during the proof of Prop.2.9 of [37] in order to collapse successively each of the pairs (x_1, x_7) , (x_2, x_6) , respectively (x_3, x_5) of critical points respectively to only one critical point and obtain in this way the desired function.

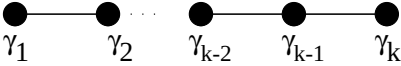
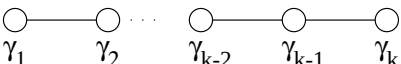
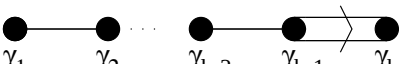
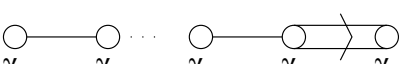
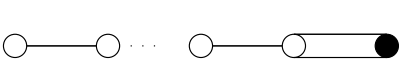
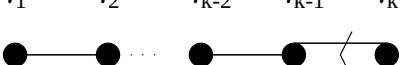
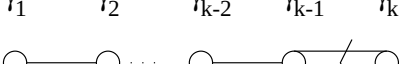
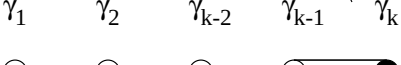
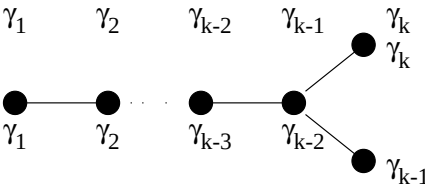
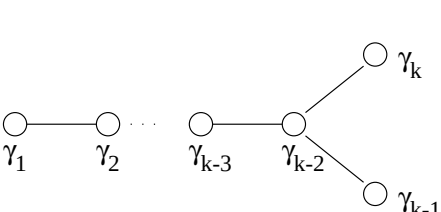
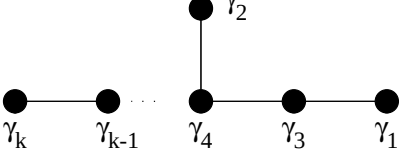
We conclude that $\text{cat}(M) \in \{4, 5\}$ and meantime we conjecture that, in fact, $\text{cat}(M) = 4$.

APPENDIX

TABLE 1 : The marked Dynkin diagrams for irreducible symmetric spaces

A_k	$\overset{m}{\circ} - \overset{m}{\circ} \dots \overset{m}{\circ} - \overset{m}{\circ} - \overset{m}{\circ}$	$m \in \{1, 2, 4\}$ or $k = 2$ and $m = 8$
B_k	$\overset{m_1}{\circ} - \overset{m_1}{\circ} \dots \overset{m_1}{\circ} - \overset{m_1}{\circ} \overset{m_2}{\circ}$	$m_1 = m_2 = 2$ or $m_1 = 1, m_2$ arbitrary
C_k	$\overset{m_1}{\circ} - \overset{m_1}{\circ} \dots \overset{m_1}{\circ} - \overset{m_1}{\circ} \overset{m_2}{\circ}$	$m_1 = m_2 \in \{1, 2\}$ or $m_1 = 2, m_2 = 1$ or $m_1 = 4, m_2 = 3$ or $m_1 = 4, m_2 = 1$
$(BC)_k$	$\overset{m_1}{\circ} - \overset{m_1}{\circ} \dots \overset{m_1}{\circ} - \overset{m_1}{\circ} \overset{m_2}{\circ}$	$m_1 = 2, m_2 = 2r + 1$ or $m_1 = 4,$ $m_2 = 4r + 3$ or $m_1 = 4, m_2 = 5$ or $k = 2$ and $m_1 = 6, m_2 = 9$
D_k	$\overset{m}{\circ} - \overset{m}{\circ} \dots \overset{m}{\circ} - \overset{m}{\circ} \begin{array}{l} \nearrow \overset{m}{\circ} \\ \searrow \overset{m}{\circ} \end{array}$	$m \in \{1, 2\}$
E_k	$\overset{m}{\circ} - \overset{m}{\circ} \dots \overset{m}{\circ} \begin{array}{c} \uparrow \overset{m}{\circ} \\ \downarrow \end{array} - \overset{m}{\circ} - \overset{m}{\circ}$	$k \in \{6, 7, 8\}$ and $m \in \{1, 2\}$
F_4	$\overset{m_1}{\circ} - \overset{m_1}{\circ} \overset{m_2}{\circ} - \overset{m_2}{\circ}$	$m_1 = m_2 \in \{1, 2\}$ or $m_1 = 1, m_2 \in \{2, 4, 8\}$
G_2	$\overset{m}{\circ} \equiv \overset{m}{\circ}$	$m \in \{1, 2\}$

TABLE 2: The first coloured Dynkin diagrams

A_k I	
A_k II	
B_k I	
B_k II	
B_k III = C_k III	
C_k I	
C_k II	
$(BC)_k$ IV = C_k IV	
D_k I	
D_k II	
E_k I $k=6,7,8$	

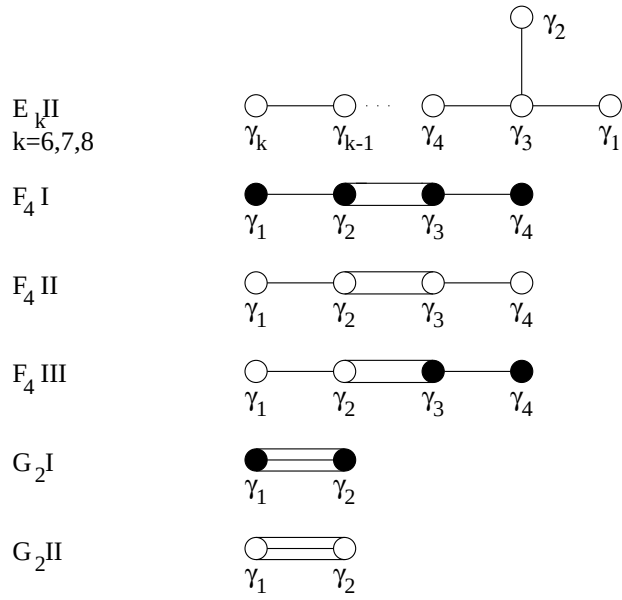
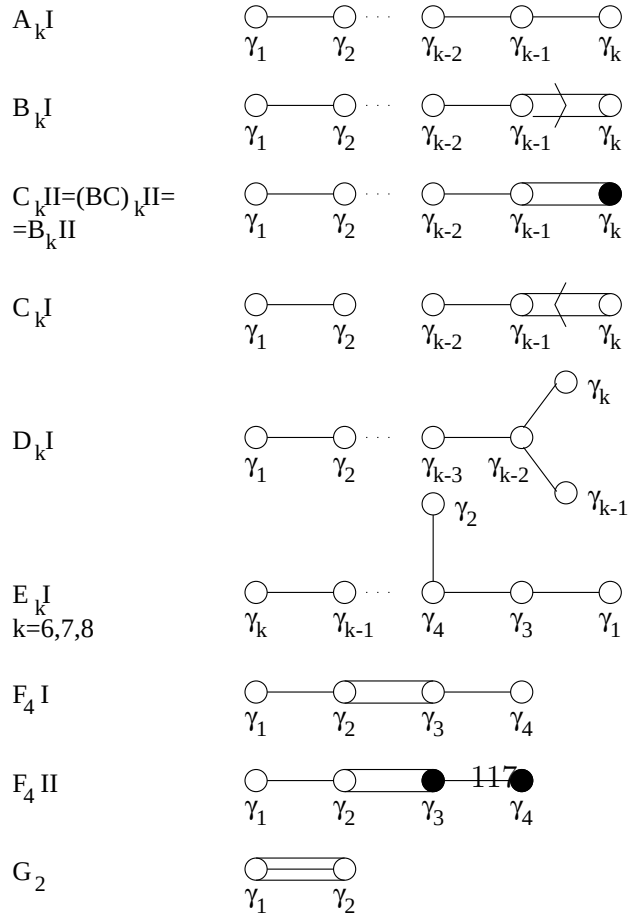


TABLE 3: The second coloured Dynkin diagrams



Bibliography

- [1] U. Abresch, *Isoparametric hypersurfaces with four or six distinct principal curvatures*, Math. Ann. **264**(1983), 283-302
- [2] M. Aubry, *Homotopy Theory and Models*, Birkhäuser (DMV Seminar, Band 24), 1995
- [3] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. Math.(2)**57**(1953), 115-207
- [4] A. Borel, *Kählerian coset spaces of semi-simple Lie groups*, Proc. Nat. Acad. Sci. **40**(1954), 1147-1151
- [5] R. Bott, H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80**(1958), 964-1029
- [6] Th. Bröcker, T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, 1985
- [7] K.S. Brown, *Buildings*, Springer-Verlag, 1989
- [8] E. Cartan, *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z. **45**(1939), 335-367
- [9] O. Cornea, *Cone-decompositions and degenerate critical points*, to appear in Proc. of the London Math. Soc.
- [10] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, 1980
- [11] J. Dorfmeister, E. Neher, *Isoparametric hypersurfaces, case $g = 6$, $m = 1$* , Commun. Algebra **13**(1985), 2299-2368

- [12] J.-H. Eschenburg, *Lecture notes on symmetric spaces*, preprint
- [13] D. Ferus, H. Karcher, H.F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math.Z. **177**(1981), 479-502
- [14] M. Greenberg, J.R. Harper, *Algebraic Topology: A First Course*, Benjamin, 1981
- [15] K. Grove, S. Halperin, *Dupin hypersurfaces, group actions and the double mapping cylinder*, J.Diff.Geom. **26**(1987), 429-459
- [16] E. Heintze, X. Liu, *Homogeneity of infinite dimensional isoparametric submanifolds*, preprint
- [17] E. Heintze, C. Olmos, *Normal holonomy groups and s-representations*, Indiana Univ. Math. J. **41**(1992), 869-874
- [18] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, 1978
- [19] W.Y. Hsiang, R. Palais, C.-l. Terng, *Topology of isoparametric submanifolds*, J.Diff.Geom. **27**(1988), 423-460
- [20] J.E. Humphreys, *Finite Reflection Groups*, Cambridge University Press, 1990
- [21] I.M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17**(1978), 331-348
- [22] R.R. Kocherlakota, *Integral homology of real flag manifolds and loop spaces of symmetric spaces*, Adv. in Math. **110**(1995), 1-46
- [23] A. Kono, K. Ishitoya, *Squaring operations in mod 2 cohomology of quotients of compact Lie groups by maximal tori*, Algebraic Topology, Barcelona 1986, Springer-Verlag (LNM 1298), 1987
- [24] S.D. Liao, *On the theory of obstructions of fiber bundles*, Ann. of Math. **60**(1), 1954, 146-191
- [25] O. Loos, *Symmetric Spaces II: Compact Spaces and Classifications*, W.A. Benjamin, New -York, 1969

- [26] J. Milnor, *Morse Theory*, Princeton University Press, 1963
- [27] M. Mimura, H. Toda, *Topology of Lie Groups*, Am.Math.Soc. (Translations of mathematical monographs), 1991
- [28] J.R. Munkres, *Elements of Algebraic Topology*, Benjamin, 1984
- [29] H.F. Münzner, *Isoparametrische Hyperflächen in Sphären I*, Math.Ann. **251**(1980), 57-71
- [30] H.F. Münzner, *Isoparametrische Hyperflächen in Sphären II*, Math.Ann. **256**(1981), 57-71
- [31] C. Olmos, *Isoparametric submanifolds and their homogeneous structure*, J.Diff.Geom. **38**(1993), 225-234
- [32] R. Palais, C.-l. Terng, *Critical Point Theory and Submanifold Geometry*, Springer-Verlag (LNM 1353), 1988
- [33] M.A. Ronan, *Buildings: main ideas and applications, I. Main ideas*, Bull.London Math.Soc. **24**(1992), 1-57
- [34] H. Seifert, W. Threlfall, *Variationsrechnung im Grossen (Theorie von Marston Morse)*, Chelsea Publishing Company, Bronx, New-York, 1971 (unaltered issue of the edition from 1938)
- [35] W. Singhof, *On the Lusternik-Schnirelmann category of Lie groups*, Math.Z. **145**(1975), 111-116
- [36] N.E. Steenrod, *Topology of Fiber Bundles*, Princeton University Press, 1951
- [37] F. Takens, *The minimal number of critical points of a function on a compact manifold and the Lusternik-Schnirelmann category*, Inventiones Math. **6**(1968), 197-244
- [38] C.-l. Terng, *Isoparametric submanifolds and their Coxeter groups*, J.Diff.Geom. **21**(1985), 79-107
- [39] G. Thorbergsson, *Dupin hypersurfaces*, Bull.London Math.Soc. **15**(1983), 493-498

- [40] G. Thorbergsson, *Isoparametric foliations and their buildings*, Ann. of Math. **133**(1991), 429-446
- [41] H. Toda. T. Watanabe, *The integral cohomology ring of F_4/T and E_6/T* , J.Math. Kyoto Univ. **14**(1974), 257-286

Index

- Assumption (of orientability), 23, 58
- cohomology of principal adjoint orbits, 92
 - definition, 98
 - of an adjoint orbit, 101
- coloured Dynkin diagram, 56, 109, 110
- cup-length
 - definition, 98
 - of an adjoint orbit, 101
- cycle
 - linking, 4
 - of Bott-Samelson type, 5, 15, 23, 42
 - HPT-, 16
 - F-, 22
- delimitations, 74
- dual basis in $H^*(M)$, 78
- F-category
 - definition, 99
 - of an adjoint orbit, 105
- flag manifold
 - full, 44
 - partial, 46
- focal manifold, 10, 31
- fundamental indices, 9
- gallery, 18
 - minimal, 18
 - type of, 18
- $H_{lower}(M)$, 53
- homology module
 - of M , 14
 - via homotopy of galleries, 20
 - of M_v , 33
- homotopic galleries, 19, 28
- isoparametric
 - submanifold, 7
 - foliation, 10
- isotropy
 - representation, 10, 37
 - orbit, 10
- Lusternik-Schnirelmann category
 - definition, 98
 - of an adjoint orbit, 101
 - of an isoparametric submanifold of type B_2 , 105
- marked Dynkin diagram, 108
- Morse
 - Lemma, 1
 - bad theory, 3
 - good theory, 4
 - inequalities, 5
- multiplicity
 - of an isoparametric submanifold, 8
 - of a root, 11
- the $\mathcal{R}$ -algebra $H^*(M)$, 79
 - for M of rank 2, 81, 85-91
- the $\mathcal{R}$ -algebra $H^*(M_v)$, 80
 - for M of uniform even multiplicity, 82

reduced word, 18, 30
representation induced by W , 54, 55
simply generated algebra, 72
Schubert cycle, 41, 42, 44, 48, 51
Thorbergsson, theorem of, 13
 $\mathbb{Z}_2$ -cohomology of M_v , 84

Lebenslauf

Name: Augustin-Liviu Mare
Geburtsdatum: 11.11.1967
Geburtsort: Campia-Turzii, Rumänien
Familienstand: verhairatet

Schul Ausbildung

1974-1982 Grundschule Campia-Turzii
1982-1986 Gymnasium Industriegymnasium, Campia-Turzii
1987-1991 Universität " Babes-Bolyai"-Universität von Cluj-Napoca,
Fakultät für Mathematik

9/1991 -12/1996 Assistent im Lehrstuhl für Geometrie, Fakultät für
Mathematik, "Babes-Bolyai"-Universität von Cluj-Napoca

seit 1/1997 Doktorand im Rahm des Graduiertenkollegs *Nichtliniare
Probleme in Analysis, Geometrie und Physik*, Universität von Augsburg

Ein berühmter Satz von Thorbergsson gibt eine vollständige Charakterisierung isoparametrischer Untermannigfaltigkeiten von Rang grösser als 2 in Euklidischen Räumen. Die seien die Hauptorbits von Isotropiedarstellungen der symmetrischen Räumen. Die Homologie dieser Räume lässt sich leicht durch Morse-Theorie erhalten. Das Hauptziel der vorliegenden Arbeit ist, die Kohomologie von beliebigen Isotropieorbits zu bestimmen. Man verallgemeinert auf diese Weise Ergebnisse von Bott und Samelson, bzw. Hsiang, Palais und Terng.