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AN INEQUALITY CONCERNING SYMMETRIC FUNCTIONS AND SOME APPLICATIONS

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Abstract

An inequality for symmetric continuous functions $E : I^n \to \mathbf{R}$ is proved in Theorem 1.1. and a variant for C^1- differentiable functions is given in Theorem.1.2. Some applications concerning inequalities between means or convex functions are presented in the second section.

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1. The main results

Let $I \subseteq \mathbf{R}$ be an interval and $I^n = I \times \ldots \times I(n\ times), I^n \subseteq \mathbf{R}^n$. Consider $(\mathcal{S}_n, \circ)$ the permutations grup of the set $\{1, 2, .., n\}$ acting on I^n by $\sigma x = (x_{\sigma(1)}, \ldots, x_{(\sigma(n))})$, where $x = (x_1, \ldots, x_n)$. Recall that a real-valued function $E : I^n \to \mathbf{R}$ is **symmetric** or $\mathcal{S}_n$**-invariant** if for every $x \in I^n$ the relation $E(\sigma x) = E(x)$ holds, i.e. E is constant on the $\mathcal{S}_n-$ orbits.

The main purpose of this section consists in proving of two general results on symmetric functions which will be very useful in obtaining some important inequalities.

Theorem 1. 1 *Let $I \subseteq \mathbf{R}$ be an interval and let $E : I^n \to \mathbf{R}$ be a symmetric continuous function satisfying for every $a = (a_1, \ldots, a_n) \in I^n$ with $a_1 \le a_2 \le \ldots \le a_n$ the inequality*

$$E(a_1, \ldots, a_n) \underset{(\ge)}{\le} E\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n\right) \tag{1}$$

Then for every $a = (a_1, \dots, a_n) \in I^n$ the following inequality holds

$$E(a_1, \dots, a_n) \underset{(\geq)}{\leq} E\left(\frac{a_1 + \dots + a_n}{n}, \dots, \frac{a_1 + \dots + a_n}{n}\right) \tag{2}$$

Proof. We prove by induction on k, $2 \leq k \leq n$, that for every $a = (a_1, \dots, a_n) \in I^n$ with $a_1 \leq a_2 \leq \dots \leq a_n$ the following inequality is satisfied

$$E(a_1, \dots, a_n) \underset{(\geq)}{\leq} E\left(\frac{a_1 + \dots + a_k}{k}, \dots, \frac{a_1 + \dots + a_k}{k}, a_{k+1}, \dots, a_n\right) \tag{3}$$

Taking into account the hypotheses (1) one obtains that the assertion is true for $k = 2$.

Let us suppose that (3) is verified for a fixed number $k \geq 2$ and denote

$$\alpha = \frac{a_1 + \dots + a_k}{k}, \quad \beta = a_{k+1} \tag{4}$$

Because $\alpha \leq \alpha \leq ... \leq \alpha \leq \beta \leq a_{k+1} \leq \dots \leq a_n$ it follows

$$E(\alpha, \dots, \alpha, \beta, a_{k+1}, \dots, a_n) \underset{(\geq)}{\leq} E(x_p, \dots, x_p, y_p, z_p, a_{k+2}, \dots, a_n) \tag{5}$$

where the sequences $(x_p), (y_p), (z_p)$ are defined by

$$y_{2p+1} = x_{2p+1} = x_{2p+2} = \frac{(k-1)x_{2p} + z_{2p}}{k}, p \geq 1$$

$$y_{2p} = z_{2p} = z_{2p+1} = \frac{x_{2p-1} + z_{2p-1}}{2}, \ p \geq 1$$

Put $u_p = z_{2p} = z_{2p+1}$, $p \geq 0$ and $u_0 = z_0 = z_1 = \beta$. We also denote $v_p = x_{2p-1} = x_{2p}$, $p \geq 1$. Then

$$\begin{cases} v_{p+1} = \frac{(k-1)v_p + u_p}{k}, \\ u_p = \frac{v_p + u_{p-1}}{2} \end{cases} \tag{6}$$

where $p \geq 1, v_1 = \alpha, u_0 = \beta$. From the relations (6) one obtains $k(v_{j+1} - v_j) = u_{j-1} - u_j$, $j \geq 1$. By adding these equalities for $j = 1, 2, \dots, p$, it follows $k(v_{p+1} - v_1) = u_0 - u_p$; so $k(v_{p+1} - \alpha) = \beta - u_p$. Therefore $kv_{p+1} = k\alpha + \beta - u_p$ and using the first relation of (6) one obtains

$$v_{p+1} = \frac{k-1}{2k} v_p + \frac{k\alpha + \beta}{2k}.$$

Because $0 \leq \frac{k-1}{2k} < 1$ it immediately follows that the sequence (v_p) is convergent and

$$\lim_{p \to \infty} v_p = \frac{k\alpha + \beta}{k+1}.$$

Moreover

$$u_p = kv_{p+1} - (k-1)v_p \to k\frac{k\alpha+\beta}{k+1} - (k-1)\frac{k\alpha+\beta}{k+1} = \frac{k\alpha+\beta}{k+1}$$

Using the continuity of the function E and the inequality (5) it follows for $p \to \infty$

$$E(\alpha,\dots,\alpha,\beta,a_{k+1},\dots,a_n) \underset{(\geq)}{\leq} E\left(\frac{k\alpha+\beta}{k+1},\dots,\frac{k\alpha+\beta}{k+1},a_{k+2},\dots,a_n\right)$$

Taking into account that (3) is satisfied for the fixed number k, one obtains

$$E(a_1,\dots,a_n) \underset{(\geq)}{\leq} E\left(\frac{a_1+\dots+a_{k+1}}{k+1},\dots,\frac{a_1+\dots+a_{k+1}}{k+1},a_{k+2},\dots,a_n\right)$$

and the assertion is proved by mathematical induction principle.

Theorem 1. 2 *Let $I \subseteq \mathbf{R}$ be an open interval and let $E : I^n \to \mathbf{R}$ be a symmetric C^1–differentiable function satisfying for every $a = (a_1,\dots,a_n) \in I^n$ with $a_1 \leq a_2 \leq \dots \leq a_n$ the inequality*

$$\frac{\partial E}{\partial x_1}(a) \underset{(\leq)}{\geq} \frac{\partial E}{\partial x_2}(a) \tag{7}$$

Then for every $a = (a_1,\dots,a_n) \in I^n$ the inequality (2) holds.

Proof. Applying the mean value theorem for the function E and the segment $[a,b]$ where $a = (a_1,\dots,a_n)$, $b = (\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\dots,a_n)$ one obtains a point $\xi \in (a,b)$ with the property

$$E(a_1,a_2,a_3,\dots,a_n) - E(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\dots,a_n) =$$

$$= \frac{\partial E}{\partial x_1}(\xi)(a_1 - \frac{a_1+a_2}{2}) + \frac{\partial E}{\partial x_2}(\xi)(a_2 - \frac{a_1+a_2}{2}) =$$

$$= \frac{1}{2}(a_2-a_1)(\frac{\partial E}{\partial x_2}(\xi) - \frac{\partial E}{\partial x_1}(\xi)) \underset{(\geq)}{\leq} 0$$

that is the condition (1) in Theorem 1.1 is satisfied and the desired conclusion is obtained.

Remark 1. 3. Suppose that the function $E : I^n \to \mathbf{R}$ is symmetric and continuous. To verify the condition (1) in Theorem 1.1 for E and $a = (a_1,\dots,a_n) \in I^n$, $a_1 \leq a_2 \leq \dots \leq a_n$, consider $\beta = \frac{a_1+a_2}{2}$, $\gamma = \frac{a_2-a_1}{2}$ and the function $\varphi : [0,\gamma] \to \mathbf{R}$ given by $\varphi(t) = E(\beta - t, \beta + t, a_3,\dots,a_n)$. If the function φ is decreasing (increasing) on $[0,\gamma]$ it follows

$$E(a_1,a_2,\dots,a_n) = \varphi(\gamma) \underset{(\geq)}{\leq} \varphi(0) = E(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\dots,a_n)$$

that (1) is satisfied and in this case one obtains the inequality (2).

If the function $E : I^n \to \mathbf{R}$ satisfies the hypotheses of Theorem 1. 2 then the derivative of function φ is given by $\varphi'(t) = -\frac{\partial E}{\partial x_1}(u(t)) + \frac{\partial E}{\partial x_2}(u(t)) \underset{(\geq)}{\leq} 0$ on $[0, \gamma]$, where $u(t) = (\beta - t, \beta + t, a_3, \ldots, a_n)$, consequently φ is decreasing (increasing) on $[0, \gamma]$, and (1) is verified.

Other results involving weighted-symmetric functions are given in the forthcoming authors' paper [3].

2. Applications

In this section the following standard notations will be used (see [4],[5]). For $I = (0, \infty), a = (a_1, \ldots, a_n) \in I^n$ let us consider

$$A_n(a) = \frac{a_1 + \ldots + n}{n} \qquad \text{(arithmetic mean)}$$

$$G_n(a) = \sqrt[n]{a_1 \ldots a_n} \qquad \text{(geometric mean)}$$

$$H_n(g) = \frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_n}} \qquad \text{(harmonic mean)}$$

$$M_n^{(\alpha)}(a) = \left(\frac{a_1^\alpha + \ldots + a_n^\alpha}{n} \right)^{\frac{1}{\alpha}}, \quad \alpha > 0 \quad \text{(mean of order } \alpha)$$

Application 2.1. Let

$$E(a_1, \ldots, a_n) = S_k(a_1, \ldots, a_n) = \sum_{i_1 \leq \ldots \leq i_k} a_{i_1} \ldots a_{i_k}$$

be the k^{th} symmetric sum of $a_1, \ldots, a_n$. It is easy to verify that for $a_1 \leq a_2 \leq \ldots \leq a_n$ the condition (7) in Theorem 1. 2. is satisfied. Therefore $S_k(a_1, \ldots, a_n) \leq S_k(A_n(a), \ldots, A_n(a))$ which is equivalent with the well-known McLaurin' inequality

$$S_k(a_1, \ldots, a_n) / \binom{n}{k} \leq (A_n(a))^k \tag{8}$$

If $k = n$, (8) becomes the arithmetic-geometric means inequality $G_n(a) \leq A_n(a)$.

Application 2.2. Consider $E(a_1, \ldots, a_n) = a_1 \ldots a_n (\frac{1}{a_1} + \ldots + \frac{1}{a_n}) = S_{n-1}(a_1, \ldots, a_n)$ satisfying for $a_1 \leq a_2 \leq \ldots \leq a_n$ the condition (7) in Theorem 1.2. Then $E(a_1, \ldots, a_n) \leq E(A_n(a), \ldots, A_n(a))$, that is $(G_n(a))^n (n/H_n(a)) \leq (A_n(a))^n (n/A_n(a))$ which represents the first part of W. Sierpinski' inequalities ([6], [5,pp. 21-25]):

$$(A_n(a))^{n-1} H_n(a) \geq (G_n(a))^n \geq A_n(a) (H_n(a))^{n-1} \tag{9}$$

Taking into account the following relations

$$A_n(\frac{1}{a}) = 1/H_n(a), G_n(\frac{1}{a}) = 1/G_n(a), H_n(\frac{1}{a}) = 1/A_n(a),$$

one obtains that the first inequality in (9) is equivalent with the second one.

The first inequality in (9) is the best in the following sense:

$$(G_n(a)/A_n(a))^\alpha \le H_n(a)/A_n(a) \iff \alpha \ge n \tag{10}$$

Using (9) one obtains that $\alpha \ge n$ is a sufficient condition for (10). To show that $\alpha \ge n$ is also necessary for (10) let us consider $a = (1-\varepsilon, 1+\varepsilon, 1, \dots, 1), \varepsilon \in [0,1)$ and it follows $(\sqrt[n]{1-\varepsilon^2})^\alpha \le \frac{n}{n-2+\frac{2}{1-\varepsilon^2}}$. Put $t = 1-\varepsilon^2$ and one obtains the equivalent inequality

$$t^{\frac{\alpha}{n}} \le \frac{n}{n-2+\frac{2}{t}}, \quad t \in (0,1]$$

therefore $(n-2)t^{\frac{\alpha}{n}} + 2t^{\frac{\alpha}{n}-1} \le n$. If $\alpha < n$, then for $t \searrow 0$ a contradiction follows and consequently $\alpha \ge n$.

Application 2.3. We shall use Theorem 1.1 to prove the inequality

$$G_n(a)M_n^{(2)}(a) \le A_n^2(a) \tag{11}$$

which is a refinement of arithmetic-geometric means inquality since $M_n^{(2)}(a) \ge A_n(a)$ (see [4,pp. 76-77).

Consider $E(a_1, \dots, a_n) = (a_1 a_2 \dots a_n)^2 (a_1^2 + \dots + a_n)^n$. Suppose $a_1 \le a_2 \le \dots \le a_n$ and put $\beta = \frac{a_1+a_2}{2}, \gamma = \frac{a_2-a_1}{2}$. Following the idea presented in Remark 1.3. let us consider the function $\varphi : [0,\gamma] \to \mathbf{R}$, $\varphi(t) = E(\beta - t, \beta + t, a_3, \dots, a_n)$. An elementary computation shows that

$$\varphi(t) = ((\beta^2 - t^2)a_3 \dots a_n)^2 (2t^2 + 2\beta^2 + a_3^2 + \dots + a_n)^2$$

and $\varphi'(t) = (a_3 \dots a_n)^2 4t(\beta^2 - t^2)(2t^2 + 2\beta^2 + a_3^2 + \dots + a_n^2)(-(n+2)t^2 + (n-2)\beta^2 - a_3^2 - \dots - a_n^2)$. Because $0 \le t \le \gamma < \beta \le a_2 \le a_3 \le \dots \le a_n$ one obtains $\varphi'(t) \le 0$ on $[0,\gamma]$, i.e. φ is decreasing on $[0,\gamma]$. Applying Remark 1.3. and Theoren 1.1 it follows (11).

The inequality (11) is strongest in the following sense:

$$(G_n(a)/A_n(a))^\alpha \le A_n(a)/M_n^{(2)}(a) \iff \alpha \ge 1 \tag{12}$$

The sufficiency of condition $\alpha \ge 1$ was proved above. For the necessity consider $a_1 = 1 + x, a_2 = 1 - x, a_3 = a_4 = \dots = a_n = 1$, where $x \in [0,1)$. Then

$$(\sqrt[n]{1-x^2})^\alpha \le \sqrt{\frac{n}{2x^2+n}}, \quad \text{thus } (1-x^2)^{\frac{\alpha}{n}}(2x^2+n)^{\frac{1}{2}} \le \sqrt{n}.$$

Let $f : [0,1] \to \mathbf{R}$ be the function given by $f(t) = (1-t)^{\frac{\alpha}{n}}(2t+n)^{\frac{1}{2}}$. Remark that for every $t \in [0,1)$, $f(t) \le f(0) = \sqrt{n}$, i.e. $t = 0$ is a maximum point of f. On the other hand the derivative of f is

$$f'(t) = -(1-t)^{\frac{\alpha}{n}-1}(2t+n)^{-1/2}((\frac{2\alpha}{n}+1)t + \alpha - 1)$$

If $\alpha < 1$, then $0 < \frac{1-\alpha}{1+\frac{2\alpha}{n}} < 1$ and one obtains that f strictly increasing on the interval $[0, \frac{1-\alpha}{1+\frac{2\alpha}{n}})$. Therefore $\sqrt{n} = f(0) < f(t),\ \ t \in (0, \frac{1-\alpha}{1+\frac{2\alpha}{n}})$ a contradiction.

Application 2.4. For a given function $g : I \to \mathbf{R}$ let us denote

$$D_g^{(n)}(a_1, \ldots, a_m) = \frac{1}{n}\sum_{i=}^{n} g(a_i) - g(\frac{1}{n}\sum_{i=1}^{n} a_i),$$

where $a_1, \ldots, a_n \in I$.

Definition. The function $f : I \to \mathbf{R}$ is m-g-**convex** if for all $a_1, a_2 \in I$ the following inequality is verified:

$$\frac{f(a_1) + f(a_2)}{2} - f(\frac{a_1 + a_2}{2}) \geq m \cdot D_g^{(2)}(a_1, a_2) \tag{13}$$

The function $f : I \to \mathbf{R}$ is M-g-**concave** if for all $a_1, a_2 \in I$ the following relation holds:

$$M \cdot D_g^{(2)}(a_1, a_2) \geq \frac{f(a_1) + f(a_2)}{2} - f(\frac{a_1 + a_2}{2}) \tag{14}$$

Let $f : I \to \mathbf{R}$ be a m-g-convex and M-g-concave continuous function on I, where $g : I \to \mathbf{R}$ is continuous and convex, $M > m$. Consider

$$E_1(a_1, \ldots, a_n) = \sum_{i=1}^{n} f(a_i) - m\sum_{i=1}^{m} g(a_i),$$

$$E_2(a_1, \ldots, a_n) = M\sum_{i=1}^{n} g(a_i) - \sum_{i=1}^{n} f(a_i).$$

It is clear that the functions $E_1, E_2 : I^n \to \mathbf{R}$ are symmetric, continuous and taking into account (13), (14) it follows that E_1, E_2 satisfy the condition (1) in Theorem 1.1. with "$\geq$". From (2) one obtains

$$M \cdot D_g^{(n)}(a_1, \ldots, a_n) \geq \frac{1}{n}\sum_{i=1}^{n} f(a_i) - f(\frac{1}{n}\sum_{i=1}^{n} a_i) \geq m \cdot D_g^{(n)}(a_1, \ldots, a_n) \tag{15}$$

which represent refinements of the well-known Jensen' inequality.

An interesting situation sudied in [1], [2] (see also [5, pp 564-566]) is given by the convex function $g : I \to \mathbf{R}, g(t) = t^2$. In this case

$$D_g^{(n)}(a_1, \ldots, a_n) = \frac{1}{n^2}\sum_{i<j}(a_i - a_j)^2$$

and if $I = [\alpha, \beta]$, then every function $f \in C^2[\alpha, \beta]$ is m-g-convex and M-g-concave on I, where

$$m = \frac{1}{2}\min\{f''(t) : t \in [\alpha, \beta]\} \text{ and } M = \frac{1}{2}\max\{f''(t) : t \in [\alpha, \beta]\}.$$

The inequalities (15) becomes

$$\frac{M}{n^2}\sum_{i<j}(a_i - a_j)^2 \geq \frac{1}{n}\sum_{i=1}^{n} f(a_i) - f(\frac{1}{n}\sum_{i=1}^{n} a_i) \geq \frac{m}{n^2}\sum_{i<j}(a_i - a_j)^2 \qquad (16)$$

which have many interesting applications (see [1] fore instance).

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