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ON THE CRITICAL SET OF A MORSE FUNCTION

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ABSTRACT. We show that if M is a closed manifold then a (finite) subset A of M is the critical set of a Morse function on M if and only if the cardinality of A is at the same time congruent mod 2 with the Euler-Poincaré characteristic of M and at least equal to the Morse number of M.

The main purpose of Morse theory is the topological study of a differentiable manifold[1] by means of its real functions. The goal is completely realized, at least from the homotopy point of view, by using the so-called Morse functions: they allow the complete description of the manifold up to homotopy type, as a CW complex, the description depending on the critical set of the function (see for instance [3]).

In a natural way, Andrica poses in [1] the following problem:

Problem. Given a closed manifold M, describe all $A \subset M$ such that A is a critical set of a Morse function on M.

The goal of this note is to completely solve this problem.

1. Preliminaries and statement of the main result

Let M be a closed (i.e. connected, compact, and without boundary), n-dimensional manifold. Denote by $\mathcal{F}(M)$ the algebra of all C^∞ real functions on M. Let f be such a function. A point p in M is a critical point of f if the differential df_p is identically zero. Denote by $C(f)$ the set of all these points, which will be called the critical set of f. We say that $p \in C(f)$ is non-degenerate if the Hessian matrix of f at p is non-singular. Being symmetric, the eigenvalues of the Hessian matrix are all real. The index of p as a critical point of f is, by definition, the number of negative eigenvalues of this matrix.

Definition 1.1. A function $f \in \mathcal{F}(M)$ is a *Morse function* if its critical points are all non-degenerate. Denote by $\mathcal{F}_m(M) \subset \mathcal{F}(M)$ the set of all Morse functions.

Fix f a Morse function on M. By using the Morse Lemma, one can easily deduce that the critical points are isolated in M; since the latter is compact, $C(f)$ is finite. One can partition $C(f)$ according to the index of each critical point. Let $c_i(f)$ be the number of critical points of index i, for all $0 \le i \le n$, and $c(f) = c_0(f) + \cdots + c_n(f)$ the total number of such points.

Also recall that the Betti numbers of M relative to a field F are defined in terms of the singular homology of M as

$$\beta_i = \beta_i(M;F) = \mathrm{rank}_F H_i(M;F).$$

[1]All manifolds here are of class C^∞.

The Euler-Poincaré chracteristic of M is

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \beta_i.$$

The following well-known result establishes a relationship between $\mathrm{Crit}(f)$ and the topology of M (see e.g. [3]):

Theorem 1.2. *If $f \in \mathcal{F}_m(M)$ then*

$$\sum_{i=0}^{q} (-1)^i c_{q-i}(f) \geq \sum_{i=0}^{q} (-1)^i \beta_{q-i}, \text{ for all } 0 \leq q \leq n-1$$
$$\sum_{i=0}^{n} (-1)^i c_i(f) = \chi(M).$$

One can also associate to the manifold M the number

$$\gamma(M) = \min\{\mathrm{card}(C(f)) \mid f \in \mathcal{F}_m(M)\},$$

called the *Morse number* or *the Morse-Smale characteristic* of M. For more insight about this invariant, we refer to [6], [2], [5], and [1].

It is now obvious that if $f \in \mathcal{F}_m(M)$ then

$$c(f) \equiv \chi(M) \bmod 2 \quad \text{and} \quad c(f) \geq \gamma(M).$$

The main result of this note is:

Theorem 1.3. *A finite subset $A \subset M$ is a critical set of a Morse function on M if and only if the cardinality of A is congruent mod 2 with $\chi(M)$ and at least equal to $\gamma(M)$.*

2. Proof of the main result

The first step is to point out that if A is a finite subset of M, then the cardinality of A alone is responsible whether A is a critical set of a Morse function. The reason is given by the following well-known result (cf. e.g. [4]):

Lemma 2.1. *If $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ are finite subsets of M (with the same cardinality), then there exists a diffeomorphism of M which transforms a_i into b_i, for all $1 \leq i \leq m$.*

Let us now define as follows:

Definition 2.2. A positive integer m is *non-degenerate critical cardinality* of M if there exists a Morse function on M with exactly m critical points.

Thus the goal is to compute all possible such integers.

We start with the following lemma:

Lemma 2.3. *There exist real numbers $a, k, \ell > 0$ with*

$$a < 1, \qquad \ell < k, \qquad \sqrt{\ell} > 1 + \sqrt{1-a},$$

and a C^∞ function $u : (-a, \infty) \to \mathbb{R}$ such that:

(1) $u(t) = \ln(t + a)$ *for all* $-a < t \leq \ell$;
(2) $u(t) = \ln(k + a)$ *(constant) for* $t \geq k$;
(3) $u'(t) > 0$ *for all* $-a < t < k$;
(4) $u'(t) < \dfrac{1}{2\sqrt{t}}$ *for all* $\ell < t < k$.

Proof. Let $\rho : (0, \infty) \to (0, \infty)$ be the standard flat cut-off function

$$\rho(s) = \begin{cases} \exp(-1/s) & s > 0, \\ 0 & s \leq 0, \end{cases}$$

and set

$$\Theta(s) = \frac{\rho(s)}{\rho(s) + \rho(1-s)}, \qquad s \in \mathbb{R}.$$

Then $\Theta \in C^\infty(\mathbb{R})$, $\Theta(s) = 0$ for $s \leq 0$, $\Theta(s) = 1$ for $s \geq 1$, Θ is increasing on $[0, 1]$, and $M := \sup_{s\in\mathbb{R}} |\Theta'(s)|$ is finite and positive.

Fix any $a \in (0, 1)$. Choose $\ell > 0$ so large that

$$\sqrt{\ell} > 1 + \sqrt{1-a}.$$

This inequality is equivalent to

$$\frac{1}{t+a} < \frac{1}{2\sqrt{t}} \qquad \text{for all } t \geq \ell,$$

which follows from the fact that

$$\frac{1}{t+a} < \frac{1}{2\sqrt{t}} \Leftrightarrow t + a > 2\sqrt{t} \Leftrightarrow (\sqrt{t} - 1)^2 > 1 - a.$$

Define the positive number

$$\delta_0 := \frac{1}{2\sqrt{\ell}} - \frac{1}{\ell + a}.$$

From the previous considerations, $\delta_0 > 0$.

By increasing ℓ we even have

$$(*) \qquad \frac{M}{\ell + a} \leq \frac{\delta_0}{2}.$$

This is possible since the inequality is equivalent to

$$4M + 2 \leq \frac{\ell + a}{\sqrt{\ell}},$$

the right-hand side having infinite limit as $\ell \to \infty$.

For $k > \ell$ define

$$\varphi_{k,\ell}(t) = \Theta\Big(\frac{t-\ell}{k-\ell}\Big), \qquad t \in \mathbb{R}.$$

Then $\varphi_{k,\ell} \in C^\infty(\mathbb{R})$, $\varphi_{k,\ell}(t) = 0$ for $t \le \ell$, $\varphi_{k,\ell}(t) = 1$ for $t \ge k$, and

$$\sup_{t\in\mathbb{R}} |\varphi'_{k,\ell}(t)| = \frac{M}{k-\ell}.$$

Again for $k > \ell$ define

$$\delta(k) := \min_{t\in[\ell,k]} \Big\{\frac{1}{2\sqrt{t}} - \frac{1}{t+a}\Big\}.$$

We clearly have

$$\lim_{k\to\ell} \delta(k) = \delta_0.$$

Hence there exists $\varepsilon > 0$ such that for all $k \in (\ell, \ell+\varepsilon)$

$$\delta(k) > \frac{\delta_0}{2}. \tag{**}$$

Recall that for any $x \in \mathbb{R}$, $x > 1$, we have $\ln(x) < x - 1$. Thus, for any $k > \ell$

$$\ln\frac{k+a}{\ell+a} < \frac{k-\ell}{\ell+a}.$$

Consequently,

$$\frac{M}{k-\ell}\ln\frac{k+a}{\ell+a} \;<\; \frac{M}{k-\ell}\cdot\frac{k-\ell}{\ell+a} = \frac{M}{\ell+a}.$$

By choice (*) of ℓ we have $\dfrac{M}{\ell+a} \le \dfrac{\delta_0}{2}$. Combining this with (**) we obtain: for every $k \in (\ell, \ell+\varepsilon)$,

$$\frac{M}{k-\ell}\ln\frac{k+a}{\ell+a} < \frac{M}{\ell+a} \le \frac{\delta_0}{2} < \delta(k). \tag{***}$$

Fix now $k \in (\ell, \ell+\varepsilon)$ and write $\varphi := \varphi_{k,\ell}$. Define

$$u(t) = (1-\varphi(t))\ln(t+a) + \varphi(t)\ln(k+a), \qquad t \in (-a,\infty).$$

Because φ is C^∞ and constant outside $[\ell,k]$, the function u is C^∞ on $(-a,\infty)$ and satisfies

$$u(t) = \ln(t+a) \;\text{ for } t \le \ell, \qquad u(t) = \ln(k+a) \;\text{ for } t \ge k.$$

Hence properties (1) and (2) from the lemma are satisfied.

Differentiate to obtain, for all $t \in (-a,\infty)$,

$$u'(t) = (1-\varphi(t))\frac{1}{t+a} + \varphi'(t)\big(\ln(k+a) - \ln(t+a)\big).$$

On $(-a,\ell)$ we have $\varphi \equiv 0$, so $u'(t) = 1/(t+a) > 0$. On (k,∞) we have $\varphi \equiv 1$ and $\varphi' \equiv 0$, so u' vanishes there (consistent with u being constant for $t > k$). For $t \in (\ell,k)$ we use that $\varphi' \ge 0$ (since Θ is strictly increasing on $(0,1)$) and $\ln(k+a) - \ln(t+a) \ge 0$, hence both summands are nonnegative; moreover the first summand is strictly positive because

$1/(t+a) > 0$ and $1-\varphi(t) > 0$ on (ℓ, k). Therefore $u'(t) > 0$ on $(-a, k)$, proving property (3).

We will now prove property (4). For $t \in (\ell, k)$ we have the decomposition of $u'(t)$ above. Recall that

$$\delta(k) = \min_{t\in[\ell,k]} \left(\frac{1}{2\sqrt{t}} - \frac{1}{t+a} \right) > 0$$

Therefore to ensure

$$u'(t) < \frac{1}{2\sqrt{t}} \quad \text{for all } t \in (\ell, k),$$

it suffices to prove

(****) $$\varphi'(t)\big(\ln(k+a) - \ln(t+a)\big) < \delta(k) \qquad \text{for all } t \in (\ell, k).$$

Because then we can say:

$$(1-\varphi(t))\frac{1}{t+a} + \varphi'(t)\big(\ln(k+a) - \ln(t+a)\big) < \frac{1}{t+a} + \varphi'(t)\big(\ln(k+a) - \ln(t+a)\big) <$$
$$< \frac{1}{t+a} + \delta(k) \leq \frac{1}{t+a} + \left(\frac{1}{2\sqrt{t}} - \frac{1}{t+a}\right) = \frac{1}{2\sqrt{t}}.$$

To prove (****), note that for $t \in [\ell, k]$ we have

$$\varphi'(t)\big(\ln(k+a) - \ln(t+a)\big) \leq \sup_{s\in[\ell,k]} |\varphi'(s)| \cdot \ln\frac{k+a}{\ell+a} = \frac{M}{k-\ell}\ln\frac{k+a}{\ell+a}.$$

By the choice of k (see (***)) the right-hand side is $< \delta(k)$. Thus the required inequality holds for all $t \in (\ell, k)$, proving property (4). □

Take now $f : \mathbb{R}^n \to \mathbb{R}$ $f(x) = x^1 + u(\|x\|^2) - \ln(k+a)$, for all $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, where u is the function constructed in the previous lemma.

Lemma 2.4. *(a) $f(x) = x^1$ for all $x = (x^1, \ldots, x^n)$ outside the open ball $B(0, \sqrt{k})$.*

(b) f has exactly two critical points, both non-degenerate and contained in the ball $B(0, \sqrt{\ell})$.

Proof. Only item (b) needs to be justified. Let $r^2 = \|x\|^2 = \sum_{i=1}^n (x^i)^2$. Then

$$\nabla f(x) = \big(1 + 2x_1 u'(r^2),\ 2x_2 u'(r^2), \ldots, 2x_n u'(r^2)\big).$$

A point x is critical iff each component is zero. If $r^2 \geq k$ then $u'(r^2) = 0$ and the first component cannot be zero. Consequently, $r^2 < k$. But then $u'(r^2) > 0$ and the vanishing of the last $n-1$ components in ∇f implies

$$x^2 = \cdots = x^n = 0.$$

Thus, all critical points lie on the x_1-axis with $r^2 = x_1^2 < k$, and must satisfy

(*) $$1 + 2x_1 u'(x_1^2) = 0.$$

Let $t = (x^1)^2$, a number lying in $[0, k)$; then $x^1 = \pm\sqrt{t}$, and the equation becomes

$$1 \pm 2\sqrt{t}\, u'(t) = 0.$$

Since $u'(t) > 0$, the "+" sign never gives a solution. Consequently, $x_1 = -\sqrt{t}$ and

$$2\sqrt{t}\, u'(t) = 1.$$

If $\ell < t < k$, then $u'(t) < \frac{1}{2\sqrt{t}}$, which makes the equation above impossible. This implies that $t \leq \ell$, thus

$$u'(t) = \frac{1}{t+a},$$

and the equation becomes

$$\frac{2\sqrt{t}}{t+a} = 1,$$

equivalent to

$$t - 2\sqrt{t} + a = 0.$$

The roots are

$$\sqrt{t} = 1 \pm \sqrt{1-a}.$$

In both cases t is indeed strictly smaller than ℓ, as indicated in the previous lemma. We conclude that the two critical points are both of the type $(x^1, 0, \ldots, 0)$, where

$$x^1 = -1 \pm \sqrt{1-a}.$$

Let us now compute the Hessian of f at these points. This is in each case an $n \times n$ diagonal matrix, whose diagonal entries except the first one are all equal to

$$2u'((x^1)^2) = \frac{2}{(x^1)^2 + a} \neq 0,$$

whereas the first diagonal entry is

$$2u'((x^1)^2) + 4(x^1)^2 u''((x^1)^2) = \frac{2}{(x^1)^2 + a} - \frac{4(x^1)^2}{((x^1)^2 + a)^2} = \frac{2(a - (x^1)^2)}{((x^1)^2 + a)^2}.$$

These expressions are both non-zero, their sign being given by

$$a - (1 \pm \sqrt{1-a})^2 = 2(a - 1 \mp \sqrt{1-a}).$$

One easily sees that exactly one of these two numbers is positive. □

Thorem 1.2 above is a direct consequence of the following proposition:

Proposition 2.5. *Let M be a closed manifold. A positive integer m is a non-degenerate critical cardinality for M if and only if:*

- $m \geq \gamma(M)$;
- $m \equiv \chi(M)$ *mod 2.*

In other words, the set of non-degenerate critical cardinalities of M consists of all integers of the type $\gamma(M) + 2r$, where r is a non-negative integer.

Proof. It is sufficient to show that if $F : M \to \mathbb{R}$ is a C^∞ function which has exactly m critical points, all of them non-degenerate, then there exists a C^∞ function $F_1 : M \to \mathbb{R}$ which has exactly $m+2$ critical points, all of them non-degenerate.

Let n be the dimension of M as a manifold. Since F has finitely many critical points, there exists a point $p \in M$ with $dF_p \neq 0$. By the local submersion theorem, there exists a coordinate chart

$$\varphi : U \xrightarrow{\cong} B(0,R) \subset \mathbb{R}^n,$$

for some $R > 0$, such that

$$F \circ \varphi^{-1}(x) = x^1 \quad \text{for all } x = (x^1, \dots, x^n) \in B(0,R).$$

Pick a number R' with $0 < R' < R$.

Define a new function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f}(y) := \frac{R'}{\sqrt{k}}\, f\left(\frac{\sqrt{k}}{R'} y\right).$$

If $|y| \geq R'$ then $\left|\frac{\sqrt{k}}{R'} y\right| \geq \sqrt{k}$, hence

$$f\left(\frac{\sqrt{k}}{R'} y\right) = \left(\frac{\sqrt{k}}{R'} y\right)^1.$$

Thus

$$\tilde{f}(y) = \frac{R'}{\sqrt{k}} \cdot \frac{\sqrt{k}}{R'}\, y^1 = y^1.$$

Therefore

$$\tilde{f}(y) = y^1 \quad \text{for all } y \notin B(0,R').$$

If x_0 is a critical point of f, then its rescaled point

$$y_0 = \frac{R'}{\sqrt{k}} x_0$$

is a critical point of $\tilde{f}$. Since the two critical points of f lie in $B(0,\sqrt{\ell})$ and are non-degenerate, the two critical points of $\tilde{f}$ lie in the ball $B\Big(0, \frac{R'}{\sqrt{k}}\sqrt{\ell}\Big) \subset B(0,R')$ and are non-degenerate as well.

Define $F_1 : M \to \mathbb{R}$ by

$$F_1(q) = \begin{cases} \tilde{f}(\varphi(q)), & q \in U = \varphi^{-1}(B(0,R)), \\ F(q), & q \notin U. \end{cases}$$

Note that for any $q \in \varphi^{-1}(B(0,R) \setminus B(0,R'))$ we have

$$\tilde{f}(\varphi(q)) = F(q),$$

since they are both equal to the first component of the vector $\varphi(q)$. Therefore F_1 is a smooth function on all of M.

To determine the critical points of F_1, take into account:

- Outside U, the functions F_1 and F agree, so F_1 has the same m critical points as F.
- Inside U, the function F_1 corresponds in coordinates to $\tilde{f}$, which has exactly two critical points, both non-degenerate.

Thus F_1 has in total exactly $m + 2$ critical points, all of them non-degenerate.

□

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