A TOPOLOGICAL PROPERTY OF THE EXPONENTIAL IMAGE

Liviu Mare

1. Preliminaries. If G is a Lie group, $\mathfrak{g}$ its Lie algebra and $\exp: \mathfrak{g} \longrightarrow G$ the exponential mapping, denote by E_G the image of this map. Many authors considered this set, obtaining algebraic descriptions for it (see for example, Lai [3], [4], [5] or Doković [1]). One may expect to deduce now some topological properties for E_G. The purpose of the present paper is that, by using the algebraic characterisation given by Lai in [3], to deduce that $\overline{\text{int } E_G} = \overline{E_G}$, first for G real, semisimple with finite center and then for $GL(n, \mathbb{R})$ (for any subspaces A of G, $\overline{A}$ always represents the closure of A in G).

The following result of Lai, [3, p.323] will be a basic tool for us:

1.1. Theorem. Let G be a connected real semisimple Lie group with finite center. Then we can find a positive integer p such that $g^p \in E_G$ for any $g \in G$.

We shall call a such positive integer p a sufficient exponent for G. Also recall, from the same work [3], that if p is a sufficient exponent for the adjoint group $\text{Int}(\mathfrak{g})$ and r the order of the center of G then rp is a sufficient exponent for G.

In investigating the topological structure of E_G, we are first interested if it is perhaps open or closed.
In the beginning we shall see that both questions admit a nega-

tive answer, even for a connected semisimple G. Our counter-example will be $SL(2, \mathbb{R})$.

A well-known result says that a matrix $A \in SL(2, \mathbb{R})$ belongs to $\exp(sl(2, \mathbb{R}))$ if and only if $A = B^2$, with $B \in SL(2, \mathbb{R})$.

To prove that $\exp(sl(2, \mathbb{R}))$ is not open, consider

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \exp\begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}$$

while for any $\varepsilon > 0$ the matrix $\begin{pmatrix} -1 & 0 \\ \varepsilon & -1 \end{pmatrix}$ does not belong to exponential image. Our main result (Th. 2.5) is intended somehow to repair this drawback.

Let observe now that $\exp(sl(2, \mathbb{R}))$ is not closed. To show this, choose $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ from its complement [7, ex.5, ch. III]. Using the up-mentioned characterisation of $\exp(sl(2, \mathbb{R}))$, one can easily deduce that

$$\begin{pmatrix} -1 & -1 \\ \varepsilon & -1 \end{pmatrix} \in \exp(sl, (2, \mathbb{R})), \quad \text{for any } \varepsilon > 0.$$

2. Main result. We begin with three technical results:

2.1. Lemma. Let $A \in GL(n, \mathbb{R})$ a non-singular matrix and $q \geq 2$ an integer. The linear operator

$$\Phi_A : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$$
$$X \longrightarrow A^{q-1} X + A^{q-2} X A + \dots + X A^{q-1}$$

is an isomorphism if and only if for any $\lambda, \mu \in \sigma(A)$ and any $\varepsilon \in \Sigma_q \setminus \{1\}$, $\lambda \neq \varepsilon\mu$. Equivalently

$$\det \prod_{\varepsilon \in \Sigma_q \setminus \{1\}} P_A(\varepsilon A) \neq 0.$$

Here $\Sigma_q = \{z \in \mathbb{C} : z^q = 1\}$, $\sigma(A)$ denotes the set of proper values of the operator A and P_A is the characteristic polynomial of A. For the proof of this Lemma please see [6].

2.2. Lemma. Let X, Y be topological spaces and f a local homeomorphism of X into Y. If A is any subspace of Y, $f^{-1}(\bar{A}) = \overline{f^{-1}(A)}$.

2.3. Lemma. Let X be a topological space, Y an open subspace of X and A a subspace of Y. Then:

(i) $(\bar{A})_Y = (\bar{A})_X \cap Y$

(ii) $(\text{int}\, A)_Y = (\text{int}\, A)_X$.

These last two lemmas are immediate results of general topology.

Let now G be a real connected semisimple Lie group with finite center, $\mathfrak{g}$ its Lie algebra, $\exp : \mathfrak{g} \longrightarrow G$ the exponential mapping and $E_G = \exp(\mathfrak{g})$. Denote by G_1 the adjoint group $\text{Int}(\mathfrak{g})$ of $\mathfrak{g}$, which is clearly a topological Lie subgroup of $GL(\mathfrak{g})$ [2, Th. 2.10., Ch. II]. If p is a sufficient exponent for G_1 and $r = |Z(G)|$ then $q = pr$ will be a sufficient exponent for G (and also for G_1). Consider the maps $f(g) = g^q$ of G into G and $f_1(x) = x^q$ of G_1 into G_1. The set

$$Q_1 = \{ g \in \text{Int}(\mathfrak{g}) : \lambda \neq \varepsilon\mu \text{ for any } \lambda, \mu \in \sigma(g) \text{ and } \varepsilon \in \Sigma_q \setminus \{1\} \}$$

is an open subspace of G_1. Being the complement of the zero set of an analytic function on G_1, is also dense in G_1.

2.4. __Proposition.__ The restriction $f_1 | Q_1$ is a local diffeomorphism of Q_1 into Q_1.

__Proof.__ For an arbitrary $g \in Q_1$, consider the map $h = L_{g^{-q}} \circ f \circ L_g : G_1 \longrightarrow G_1$ (L_g denotes the left translation by g). It will be sufficient to prove that h is a local diffeomorphism at the identity I of G_1. Because G_1 is a submanifold of $GL(\mathfrak{g})$, we can calculate the differential of h at I by means of the Gâteaux differential.

So $(dh)_I : ad\mathfrak{g} \longrightarrow ad\mathfrak{g}$ is given by

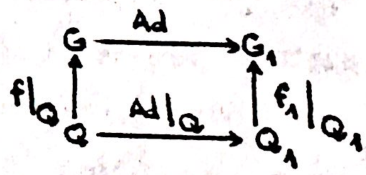
$$(dh)_I(A) = \lim_{\lambda \to 0} \frac{1}{\lambda}[h(I+\lambda A) - h(I)] =$$

$$= g^{-q+1}(A g^{q-1} + g A g^{q-2} + \dots + g^{q-1} A).$$

Finally Lemma 1.1, completes the proof.

Consider now $Ad: G \longrightarrow G_1$ the adjoint representation of G, which is clearly a local diffeomorphism. The set $Q = Ad^{-1}(Q_1)$ is an open subspace of G. By Lemma 2.2, Q is also dense in G.

Because $f_1 | Q_1$ and Ad are local diffeomorphisms, the same is true for $f | Q$.



So $f(Q)$ is an open subset of G, $f(Q) \subseteq f(G) = E_G$.
On the other hand, $E_G = f(G) = f(\overline{Q}) \subseteq \overline{f(Q)}$.
By $f(Q) \subseteq E_G \subseteq \overline{f(Q)}$ one can immediatly conclude:

2.5. __Theorem.__ For any real connected semisimple Lie group G with finite center we have

$$\overline{E_G} = \overline{\text{int } E_G}.$$

3. An application. In the sequel we shall use Th.2.2. to obtain a similar result for the exponential mapping on the space $M_n(\mathbb{R})$ of real square $n \times n$ matrices.

The Lie algebra of the Lie group $GL(n, \mathbb{R})$ can be identified with $M_n(\mathbb{R})$. As for $SL(n, \mathbb{R})$, its Lie algebra is the subalgebra $sl(n, \mathbb{R})$ of $gl(n, \mathbb{R}) = M_n(\mathbb{R})$. The unit component of $GL(n, \mathbb{R})$ is

$$GL^+(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) : \det A > 0 \} .$$

If $\mathbb{R}_+^*$ denotes the set of the strictly positive real numbers, the diffeomorphism

$$\varphi : \mathbb{R}_+^* \times SL(n, \mathbb{R}) \longrightarrow GL^+(n, \mathbb{R})$$
$$(t, A) \longrightarrow tA$$

maps $\mathbb{R}_+^* \times E_{SL}$ homeomorphically onto E_{GL} (we denoted by E_{GL}, E_{SL} the exponential images corresponding to $GL(n, \mathbb{R})$ respectively $SL(n, \mathbb{R})$). By using Lemma 2.3. one deduce

$$\overline{\text{int } E_{GL}} \cap GL^+(n, \mathbb{R}) = \overline{E_{GL}} \cap GL^+(n, \mathbb{R})$$

and because $GL^+(n, \mathbb{R})$ is closed (as a connected component),

$$\overline{\text{int } E_{GL}} = \overline{E_{GL}} .$$

Evidently all topological operators involved here (closure and interior) were considered relatively to the topology of $GL(n, \mathbb{R})$.

4. Final remarks. Our main result (Th.2.5.) is certainly just a first step in the topological study of the set E_G. It would be interesting for instance to see when E_G is dense in G.

All results of our 2-nd section remain valid in the complex case (that is considering complex connected semisimple Lie groups). But a result, mentioned for instance in [1] , says that in this case E_G is dense in G. So the real case seems to be more interesting from the point of view of our present paper.

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"Babeş-Bolyai" University
Faculty of Mathematics
Str. M.Kogălniceanu, 1
R-3400 Cluj-Napoca
Romania

On a linear matrix operator

Liviu Mare

Facultatea de Matematică
Universitatea din Cluj-Napoca
Str. M.Kogălniceanu, 1
3400 Cluj-Napoca

Let $A \in GL(n,\mathbb{R})$ be a non-singular matrix and $q \geq 2$ an integer. We stated through Lemma 2.1. of [3] a characterisation of all A for which the operator $F: M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$,

$$F(X) = X A^{q-1} + AXA^{q-2} + \dots + A^{q-1}X \qquad (1)$$

is a linear isomorphism. We shall give in this note a complete proof of this result, based on some general considerations of linear algebra.

It will be convenient for our treatment to write F as

$$F(X) = (X+AXA^{-1} + \dots + A^{q-1} XA^{-(q-1)})A^{q-1}$$

and consider the map $G: M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$,

$$G(X) = X+AXA^{-1} + \dots + A^{q-1} XA^{-(q-1)} \qquad (2)$$

To every $\mathbb{C}$ - linear endomorphism E of a certain $\mathbb{C}^P$, associate the set denoted by $\sigma(E)$ of all proper values of E. Let H be an $\mathbb{R}$- linear endomorphism of a certain space $\mathbb{R}^P$.

Define $H^{\mathbb{C}}: \mathbb{C}^P \longrightarrow \mathbb{C}^P$ by

$$H^{\mathbb{C}}(x+iy) = H(x) + i H(y)$$

for any $x, y \in \mathbb{R}^P$. One evidently obtains a $\mathbb{C}$-linear endomorphism of $\mathbb{C}^P$, which is a $\mathbb{C}$-linear isomorphism if and only if H is an $\mathbb{R}$-linear one. Define

$$\sigma(H) = \sigma(H^{\mathbb{C}}).$$

We are interested now by the proper values of the $\mathbb{C}$-linear endomorphism $G^{\mathbb{C}}: M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$,

$$G^{\mathbb{C}}(X) = X + AXA^{-1} + \dots + A^{q-1} X A^{-(q-1)} \quad (3)$$

In a natural way, consider $\varphi_A: M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$,

$$\varphi_A(X) = A X A^{-1}.$$

Lemma. The proper values of φ_A are given by

$$\sigma(\varphi_A) = \left\{ \frac{\lambda}{\mu} : \lambda, \mu \in \sigma(A) \right\}.$$

Proof. Let first be $\alpha \in \sigma(\varphi_A)$, so

$$AX - X(\alpha A) = 0,$$

for a non-zero matrix. X. By a well-known result of matrix theory (see for instance [1] p.288 or [2] p.222), α must be on the form $\frac{\lambda}{\mu}$, $\lambda, \mu \in \sigma(A)$.

Conversely, let $\lambda, \mu \in \sigma(A) = \sigma({}^tA)$ and let

$$x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

be two vectors of $\mathbb{C}^n$ such that $Ax = \lambda x$ and

$^tAy = \mu y$ or $A^{-1} = \frac{1}{\mu} \cdot {}^ty$. If $X = (x_i y_j)_{1 \le i,j \le n}$

then

$$AX = \frac{\lambda}{\mu} XA.$$

Put $\sum_q = \{ z \in \mathbb{C}: z^q = 1 \}$. We are in position to state our main result.

<u>Theorem</u>. The following conditions are equivalent.

(i) The map $F: M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ given by (1) is an $\mathbb{R}$-linear isomorphism.

(ii) The map $G^{\mathbb{C}}: M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ given by (3) is a $\mathbb{C}$-linear isomorphism.

(iii) For any $\lambda, \mu \in \sigma(A)$ and any $\varepsilon \in \sum_q \setminus \{1\}$

$\lambda \neq \varepsilon \mu$.

<u>Proof</u>. We must only observe that

$$G^{\mathbb{C}} = I + \varphi_A + \varphi_A^2 + \dots + \varphi_A^{q-1},$$

and so
$$\sigma(G^{\mathbb{C}}) = \{ \sum_{k=0}^{q-1} (\frac{\lambda}{\mu})^k : \lambda, \mu \in \sigma(A) \}.$$

A simple result of linear algebra says that $G^{\mathbb{C}}$ is an isomorphism if and only if $0 \notin \sigma(G^{\mathbb{C}})$.

<u>Example</u>. As an imediate consequence, obseve that for any $A \in GL(2, \mathbb{R})$ and any integer $q \geq 1$, the map $F: M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$,

$$F(X) = XA^{2q} + AXA^{2q-1} + \dots + A^{2q}X$$

is a linear isomorphism.

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