Math 103 Sample Midterm Questions

1. Consider the function $f(x) = \dfrac{x^3 + 3x^2 - 4x}{\sqrt{3-x}}$.

   a) Find and state the domain of f(x).
   b) Find and state all x-and y-intercepts of f(x).

2. If $f(x) = x^2 - 2x$ evaluate and simplify the expression for $f(x) - 2f(x+1) - x$

3. A rectangular storage container is to be built. It must contain 3m^3, have a square bottom and no top. If the material of the bottom costs $\$30/\text{m}^2$ and the material of the sides costs $\$15/\text{m}^2$, express the total material cost C as a function of the container's width x.

4. Find the following limits. If a given limit does not exist, state so. (If a limit is infinite, evaluate both one-sided limits.)

   a) $\displaystyle\lim_{x\to 3} \frac{9 - x^2}{x^2 - x - 6}$

   b) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3}$

   c) $\displaystyle\lim_{x\to 1} \frac{x-2}{x-1}$

5. In each case, is the given function continuous at x=1? Give a clear reason using the appropriate limits.

   a) $f(x) = \begin{cases} 2 - x^2 & \text{if} \quad x < 1 \\ 4x - 3 & \text{if} \quad x \geq 1 \end{cases}$

   b) $f(x) = \begin{cases} \dfrac{3x^2 - x - 2}{x^2 - 1} & \text{if} \quad x \neq 1 \\ 2 & \text{if} \quad x = 1 \end{cases}$

6. <u>Using the limit definition</u> of the derivative, find $f'(4)$ $\quad f(x) = \dfrac{3}{x-2}$.

7. Find the indicated derivatives using the rules of differentiation.

a) $g(t) = \frac{2}{3}t^{-2} + \frac{1}{7}t^{4/5} + 3t^{-3} - 4$ $\qquad g'(t) = ?$

b) $f(x) = \frac{x^2+1}{2-\sqrt{x}}$ $\qquad f'(1) = ?$

c) $f(x) = \sqrt[3]{x^2+1}$ $\qquad f''(x) = ?$

8. The position (in metres) of an accelerating particle after t seconds is given by the function

$$p(t) = (t+1)(2t^2+3)^3$$

How far is the particle, and at what speed is it traveling, after 6 seconds?

9. A profit function, i.e. the profit P (in thousands of \$) of producing and selling x items is given by $P(x) = -\frac{1}{300}x^2 + 7x - 100$

a) Find the profit of producing and selling 600 items.
b) Find the derivative function P'(x). What information does this function give you?
c) At what production level is P'(x)=0? What information does this point give you?

10. Find the equation of the tangent line to the curve given by $x^2y + y = 3x^3$ at the point x=1.

11. Price x (in \$) and quantity y (in thousands of units) are determined by the equation

$$3xy + 8y^2 = x^3 - 8$$

Currently, price is \$4 and quantity is 2 thousand units. If prices are rising at a rate of \$0.05 per day, how is quantity changing at this time?

12. Consider the function $f(x) = 2x^3 - 3x^2 - 12x$.

a) Find and test the critical values to determine the (x,y) locations of all local extreme points. Show all work.

b) Use your results in (a) to sketch a graph of the function.

Answers:

1.
a) We need to consider both the root and division, which combines to require that $3-x>0$,
i.e. the domain is all $x<3$

b) For the y-intercept, calculate $f(0)=0$. Hence the y-intercept is $y=0$.
For the x-intercept, solve $f(x)=0$. We only need to consider the numerator,
which factors as $x(x+4)(x-1)=0$, hence x-intercepts are $x=0$, $x=-4$ and $x=1$

2. $f(x)-2f(x+1)-x = (x^2-2x) - 2[(x+1)^2-2(x+1)]-x$
$= - x^2 -3x + 2$

3. Let the width/length of the base be x, let the height be y.
Then total volume is $V=x^2y$
Total cost is C = bottom + 4 sides
$= 30x^2 + 60 xy$ Note that we must eliminate "y"
Now use the fact that the volume is $V=3$ to solve for $y=3/x^2$
and substitute to get $C(x)=30x^2 +180/x$

4.
a) $\lim_{x\to 3} \frac{(3-x)(3+x)}{(x-3)(x+2)} = -\frac{6}{5}$

b) $\lim_{x\to 3} \frac{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}{(x-3)(\sqrt{x}+\sqrt{3})} = \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})} = \frac{1}{\sqrt{3}+3}$

c) Since this is of form "-1/0", the limit is infinite.

From the left, $\lim_{x\to 1-} \frac{x-2}{x-1} = +\infty$

From the right, $\lim_{x\to 1+} \frac{x-2}{x-1} = -\infty$

5. For a function to be continuous at x=1, we need $\lim_{x\to 1} f(x) = f(1)$.

a) 1. Check $f(1)=1$ (using 2nd line).
2. For the limit we need to check both on sided limits:
In this case both $\lim_{x\to 1-} f(x) = 1$, $\lim_{x\to 1+} f(x) = 1$ so the limit is equal to 1.
3. Both limit and function value are equal, hence the function is continuous.

b) 1. Check $f(1)=2$ (given in 2nd line)
2. Evaluate the limit: $\lim_{x\to 1} f(x) = 4$
3. The limit does not equal the function value, this f(x) is not continuous at x=1.

6. $f'(4) = \lim_{h \to 0} \dfrac{\dfrac{3}{(4+h)-2} - \dfrac{3}{4-2}}{h} = \lim_{h \to 0} \dfrac{-3}{(2+h)(2)} = -3/4$

7.

a) $g'(t) = -\dfrac{4}{3}t^{-3} + \dfrac{4}{35}t^{-1/5} - 9t^{-4}$

b) $f'(x) = \dfrac{(2x)(2-x^{1/2}) - (x^2+1)(-\frac{1}{2}x^{-1/2})}{(2-x^{1/2})^2}$ and f'(1)=3

c) $f'(x) = \dfrac{1}{3}(x^2+1)^{-2/3}(2x)$

and $f''(x) = -\dfrac{2}{9}(x^2+1)^{-5/3}(2x)^2 + \dfrac{2}{3}(x^2+1)^{-2/3}$

8. The position is given by p(6)=2953125, i.e. in 6 seconds it is 2953 km away.
It's velocity is the derivative $p'(t) = (1)(2t^2+3)^3 + (t+1)(3)(2t^2+3)^2(4t)$ hence after 6 seconds it's velocity is 3,256,875 m/sec.

(Note: this is very fast, but still only a fraction of the speed of light, which is 300,000,000 m/sec. Of course, given a little bit more time, the velocity calculated by this speed will reach that limit, so presumably this is where we would have to cut off the domain of the function. And no, you don't need to know this for the midterm. ☺)

9.

a) The profit is P(600)=2900, i.e. $2.9 million when 600 items are sold.

b) $P'(x) = \dfrac{-x}{150} + 7$ This is the "marginal profit", i.e. the rate at which profit is changing when the production size is increased.

c) Solve P'(x)=0 for x=1050. This value gives us the point at which no additional profit can be obtained by increasing production size. In other words, to attain maximum profit we should be producing 1050 items. That maximum profit is P(1050)=$3.575 million

10. Note that the equation is implicit!

To find the tangent line y=mx+b we first find m by calculating the derivative dy/dx:

$$2xy + x^2\frac{dy}{dx} + \frac{dy}{dx} = 9x^2$$

Solve for $\frac{dy}{dx} = \frac{9x^2 - 2xy}{x^2 + 1}$

Now plug in x and y to find the slope at the point. We are given x=1, to find y we need to use the original equation: $(1)^2 y + y = 3(1)^3$ and solve for y=3/2

Plug in to find that the slope m=3.

Next find the b-value by using m=3, x=1, y=3/2 in the equation y=mx+b:

3/2 = (3)(1) + b solves for b=-3/2

Hence the equation of the tangent line is y= (3/2) x - (3/2)

11. Both price and quantity are functions of time t, this is a related rates question.

Given: x=4, y=2, dx/dt=0.05 Want: dy/dt =?

Differentiate implicitly : $3\frac{dx}{dt}y + 3x\frac{dy}{dt} + 16y\frac{dy}{dt} = 3x^2\frac{dx}{dt}$

Plug in the given values and solve for dy/dt=22/440 =0.047

At this time, available quantity is rising at a rate of 47 units a day.

(Note that at this point in time the market is responding with a rise in both price and available quantity)

12.

a) y-intercept is y=0, x-intercept is x=0 and $x = \frac{3 \pm \sqrt{105}}{4}$ using quadratic formula,

i.e. about x=0, x≈3.31 and x≈-1.81

b) Find the critical points:

i) f’(x)=0 solve $6x^2$-6x-12=0 for x=2 and x=-1.

ii) f’(x) d.n.e no such points Hence we have two critical points.

Now test them (I’ll use the 2nd derivative test with f’(x)=12x-6)

f’(2)>0 hence (x,y)=(2,-20) is a local minimum

f’(-1)<0 hence (x,y)=(-1, 8) is a local maximum

c)