UNIVERSITY OF REGINA
DEPARTMENT OF MATHEMATICS & STATISTICS
Mathematics 103-001/991
Final Examination
Winter 2012

Time: 3 hours
Instructors:

Peter Douglas (Section: 001)

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NAME: ____________________

ID: ____________________

SECTION: ____________________

Show all of your work on the exam paper. Use the back of the page if necessary.

MARKS

[6] 1. Find the derivative of $f(x) = x^2$ using the limit definition of the derivative. (No marks will be given if you do not use the limit definiton.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} (2x + h) \qquad = 2x$$

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2. Find the following limits (if they exist)

[4] (a) $\lim_{x\to 5} \frac{25-x^2}{x-5}$ $= \lim_{x\to 5} \frac{(5-x)(5+x)}{(x-5)}$

$= \lim_{x\to 5} -(5+x) = -10$

[4] (b) $\lim_{x\to -3} \frac{x^2-1}{x^2+2x-3}$ $= \lim_{x\to -3} \frac{\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x-1)}}$ "$\frac{-2}{0}$"

$\rightarrow$ INFINITE LIMIT, DO BOTH SIDES

LEFT: $\lim_{x\to -3^-} \frac{x+1}{x+3} = +\infty$

RIGHT: $\lim_{x\to -3^+} \frac{x+1}{x+3} = -\infty$

[4] (c) $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$ $= \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$

$= \lim_{t\to 0} \frac{t^2+9-9}{t^2(\sqrt{t^2+9}+3)}$

$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{6}$

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[5] 3. Find the value of the constant A which makes the function continuous for all values of x.

$$f(x) = \begin{cases} 2x+3 & if x < 1 \\ Ax-1 & if x \geq 1 \end{cases}$$

NEED $f(1) = \lim_{x\to 1} f(x)$.

LEFT: $\lim_{x\to 1^-} f(x) = 5$

RIGHT: $\lim_{x\to 1^+} f(x) = A-1$

So $5 = A-1$

$\underline{\underline{A=6}}$

4. Differentiate the following functions.

[4] (a) $f(x) = x^3 \ln x$

$$f'(x) = 3x^2 \ln x + x^3 \cdot \frac{1}{x}$$

[4] (b) $f(x) = \dfrac{x+1}{x^3+5}$

$$f'(x) = \frac{(x^3+5) - (x+1)(3x^2)}{(x^3+5)^2}$$

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[4] (c) $f(x) = \dfrac{x}{e^{x^2}}$

$$f'(x) = \frac{e^{x^2} - 2x^2 e^{x^2}}{\left(e^{x^2}\right)^2}$$

[5] 5. Find $\dfrac{dy}{dx}$ if

$$x^2y - y^2x = 25$$

$$2xy + x^2\frac{dy}{dx} - 2y\frac{dy}{dx}x - y^2 = 0$$

$$\Longrightarrow \frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$$

[5] 6. Find the equation of the tangent line to the graph of $g(x) = \sqrt{5x+1}$ when $x = 3$.

FIND $y = mx + b$, $m = g'(3)$.

$$g'(x) = \frac{1}{2}(5x+1)^{-1/2}(5)$$

$$g'(3) = \frac{1}{2}(16)^{-1/2}(5) = \frac{5}{8}$$

USE POINT $(x, y) = (3, 4)$

$$\Rightarrow \quad 4 = \left(\frac{5}{8}\right)(3) + b \quad \Rightarrow \quad b = \frac{17}{8}$$

$\Rightarrow$ THE EQUATION IS $y = \frac{5}{8}x + \frac{17}{8}$

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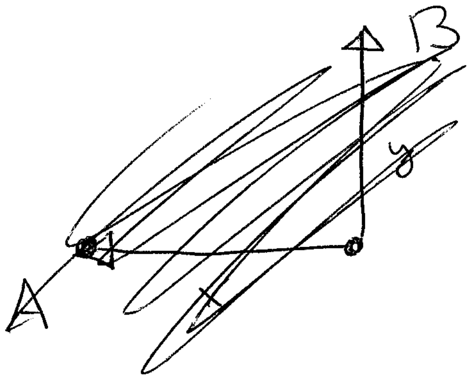
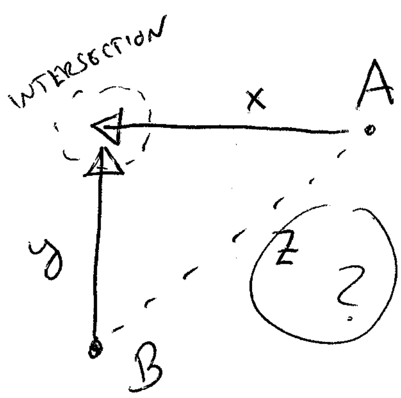
[6] 7. Car A is travelling west at 50 kilometers/hr and car B is travelling north at 60 kilometers/hr. Both are headed for the same intersection of the two roads. The roads meet at right angles. At what rate are the cars approaching each other when car A is 0.3 kilometers and car B is 0.4 kilometers from the intersection?





GIVEN: $x = 0.3$

$y = 0.4$

$z = 0.5$ (USE PYTHAGORAS)

$\frac{dx}{dt} = -50\ km/h$

$\frac{dy}{dt} = -60\ km/h$

WANT: $\frac{dz}{dt} = ?$

$\Rightarrow$ USE $x^2 + y^2 = z^2$

DIFFERENTIATE WRT TIME: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = -78\ km/h$$

THEY ARE APPROACHING EACH OTHER AT $78\ km/h$.

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[6] 8. A tumor is modeled as a sphere of radius r cm. At what rate is the volume $V = \frac{4}{3}\pi r^3$ changing with respect to r when $r = 0.75$ cm?

FIND $\cancel{\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}}$ $\frac{dV}{dr} = 4\pi r^2$

WHEN $r = 0.75$,

$$\frac{dV}{dr} = 2.25\pi$$

$$\approx 7.06 \text{ cm}^2$$

[6] 9. Determine if the function $f(x) = \dfrac{-1}{x+1}$ has an absolute maximum or an absolute minimum on the interval $x \geq 0$. Explain your answer.

NOTE: $f'(x) = \dfrac{1}{(x+1)^2}$ WHICH IS ALWAYS POSITIVE ON $x \geq 0$

SO THERE ARE NO CRITICAL POINTS

HENCE NO MAX/MIN VALUES.

10. Compute the following integrals.

[4] (a) $\int (3x^2 - \sqrt{x} + 1)\, dx$

$$= x^3 - \frac{2}{3}x^{3/2} + x + C$$

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[4] (b) $\int \frac{\ln x}{x} dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$= \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln x)^2 + C'$

[4] (c) $\int \frac{e^{1/x} - 1}{x^2} dx = \int \frac{e^{\frac{1}{x}}}{x^2} dx - \int \frac{1}{x^2} dx$

LET $u = \frac{1}{x}$

$du = -\frac{1}{x^2} dx$

$I = -\int e^u du$

$= -e^{\frac{1}{x}} + C$

$I = + x^{-1} + C$

ANSWER IS

$-e^{\frac{1}{x}} + x^{-1} + C$

[5] 11. Find the position function, $s(t)$, of an object moving in a straight line if the velocity function $v(t) = -3t^2 + 14t + 1$ and $s(0) = 2$, where t is the time measured in seconds.

POSITION $s(t) = \int v(t) \, dt$

$= -t^3 + 7t^2 + t + C'$.

FIND C', USE $s(0) = 2$

IE $2 = -0^3 + 0^2 + 0 + C$.

SO $C = 2$

$\Rightarrow$ POSITION FUNCTION

IS $s(t) = -t^3 + 7t^2 + t + 2$

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[12] 12. Consider the graph of $f(x) = 4x^3 - x^4$
Identify (i) the domain, (ii) all intercepts, (iii) all relative and absolute extreme points and (iv) all inflection points. Use your results to sketch a graph of f(x).

(i) DOMAIN : ALL $x \in \mathbb{R}$

(ii) INTERCEPTS : y-INT $y = 0$

X-INT SOLVE $0 = 4x^3 - x^4$
$= x^3(4-x)$
$\Rightarrow x=0,\ x=4$

(iii) MAX / MIN : $f'(x) = 12x^2 - 4x^3$
$f''(x) = 24x - 12x^2$

FIND CRITICAL POINTS: $f'(x) = 0$ WHEN $x=0,\ x=3$
$f'(x)$ DNE $\rightarrow$ NO SUCH POINTS

TEST :

	0		3	
$f'>0$		$f'>0$		$f'<0$
	NOT A MAX/MIN ~~MAX AT $(x,y)=(0,0)$~~		~~NOT A MAX~~/MIN MAX AT $(3, 27)$	

(iv) INFLECTION POINTS : $f''(x) = 0$ WHEN $x=0,\ x=2$
$f''(x)$ DNE $\rightarrow$ NO SUCH POINTS

TEST:

	0		2	
$f''<0$		$f''>0$		$f''<0$
	INFLECTION POINT		INFLECTION POINT	

SKETCH :