UNIVERSITY OF REGINA
DEPARTMENT OF MATHEMATICS & STATISTICS
Mathematics 103-001/991
Final Examination
Fall 2011

Time: 3 hours

Instructors:

Mahmud Manjegani (Section: 001)

Dipra Mitra (Section: 991)

NAME: ______________________

ID: ______________________

SECTION: ______________________

Show all of your work on the exam paper. Use the back of the page if necessary.

MARKS

[6] 1. Find the derivative of $f(x) = \sqrt{x}$ using the limit definition of the derivative. (No marks will be given if you do not use the limit definiton.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - (x)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

NAME: ______________________
STUDENT NO.: ______________
SECTION: ________________

2. Find the following limits (if they exist).

[4] (a) $\lim_{x\to -2} \frac{x^2-x-2}{x^2+3x+2}$ $= \lim_{x \to -2} \frac{(x-2)\cancel{(x+1)}}{(x+2)\cancel{(x+1)}}$ $\left(\text{“} = \frac{-4}{0} \text{”} \right)$

$\rightarrow$ INFINITE LIMIT, DO BOTH ONE SIDED LIMITS:

LEFT: $\lim_{x \to -2^-} \frac{x-2}{x+2} = +\infty$, RIGHT: $\lim_{x \to -2^+} \frac{x-2}{x+2} = -\infty$

[4] (b) $\lim_{x\to -1} \frac{x^2-x-2}{x^2+3x+2}$ $= \lim_{x \to -1} \frac{(x-2)\cancel{(x+1)}}{(x+2)\cancel{(x+1)}}$

$= \frac{-3}{1} = -3$

[4] (c) $\lim_{x\to -\infty} \frac{-3x^5+2x^2-1}{4x^2+5x+2}$ $= \lim_{x \to -\infty} \frac{-3 + 2/x^3 - 1/x^5}{4/x^3 + 5/x^4 + 2/x^5}$

$= -\infty$

NAME: ______________________
STUDENT NO.: ______________
SECTION: ___________________

[6] 3. For what value of the constant c is the function $f(x)$ continuous at $x = 2$.

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

NEED $f(2) = \lim_{x \to 2} f(x)$.

LEFT: $\lim_{x \to 2^-} f(x) = 4c + 4$

RIGHT: $\lim_{x \to 2^+} f(x) = 8 - 2c$

SOLVE

$4c + 4 = 8 - 2c$

$6c = 4$

$c = 2/3$

FOR $f(x)$ TO BE CONTINUOUS, WE NEED $c = 2/3$.

4. Differentiate the following functions.

[4] (a) $f(x) = \sqrt[5]{x^2 + 1}$

$$f'(x) = \frac{1}{5}(x^2+1)^{-4/5}(2x)$$

[4] (b) $f(x) = \dfrac{\ln(x^2 + 1)}{x^3 + 5x}$

$$f'(x) = \frac{\left(\frac{2x}{x^2+1}\right)(x^3+5x) - \ln(x^2+1)(3x^2+5)}{(x^3+5x)^2}$$

NAME: ______________________
STUDENT NO.: ______________
SECTION: ___________________

[4] (c) $f(x) = (x^2+1)e^{\sqrt{x}}$

$$f'(x) = 2x e^{x^{1/2}} + (x^2+1)\, e^{x^{1/2}} \cdot \left(\tfrac{1}{2} x^{-1/2}\right)$$

[6] 5. Find the equation of the tangent line to the graph of the function $g(x) = \dfrac{2x+1}{x+3}$ when $x = 0$.

FIND $y = mx + b$

WHERE $m = g'(0)$

$$g'(x) = \frac{2(x+3) - (2x+1)}{(x+3)^2} \; ; \; g'(0) = \frac{6-1}{9} = 5/9$$

FIND b, USE POINT $(x, y) = (0, 1/3)$:

$$\frac{1}{3} = \frac{5}{9} \cdot 0 + b \Rightarrow b = 1/3$$

THE EQUATION IS $y = \frac{5}{9}x + \frac{1}{3}$

NAME: ____________________
STUDENT NO.: ______________
SECTION: _________________

[6] 6. A spherical ballon is inserted into a clogged artery and is inflated at the rate of 0.002 cubic millimeters per minute. How fast is the radius of the of the ballon growing when the radius is 0.005 millimeters?

(Recall that $V = \frac{4}{3}\pi r^3$, where V is the volume of the sphere and r is the radius of the sphere)

DIFF. WRT t: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

GIVEN $\frac{dV}{dt} = 0.002$, $r = 0.005$

SOLVE FOR $\frac{dr}{dt}$:

$$0.002 = 4\pi (0.005)^2 \frac{dr}{dt}$$

THE RADIUS IS GROWING AT 6.36 mm/min

[5] 7. A medical research team determines that t days after an epidemic begins,

$$N(t) = 10t^3 + 5t + \sqrt{t}$$

people will be infected. At what rate is the infected population changing with respect to time on the ninth day?

FIND RATE OF CHANGE, IE DERIVATIVE:

$$N'(t) = 30t^2 + 5 + \frac{1}{2}t^{-1/2}$$

ON DAY 9:

$$N'(9) = 2435.167 \text{ INFECTED / DAY}$$

NAME: ____________________
STUDENT NO.: ____________________
SECTION: ____________________

8. Compute the following integrals.

[4] (a) $\int (x^3 + \sqrt{x} - \frac{1}{x^4})\ dx$

$$= \frac{1}{4} x^4 + \frac{2}{3} x^{3/2} + \frac{1}{3} x^{-3} + C$$

[4] (b) $\int \ln e^{(x^2)}\ dx$

$$= \int x^2\, dx = \frac{1}{3} x^3 + C$$

(LOG LAWS)

[4] (c) $\int 3x^2 \sqrt{x^3+1}\ dx$

$u = x^3 + 1$

$\frac{du}{dx} = 3x^2 \rightarrow du = 3x^2 dx$

$$\int u^{1/2}\, du = \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} (x^3+1)^{3/2} + C$$

NAME: ______________________
STUDENT NO.: ______________
SECTION: ___________________

[4] (d) $\int \frac{1}{2x+1}\,dx$

$u = 2x+1$

$du = 2\,dx$

$= \frac{1}{2} \int \frac{1}{u}\,du$

$= \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|2x+1| + C$

[4] (e) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$

$u = e^x - e^{-x}$

$\frac{du}{dx} = e^x + e^{-x}$

$= \int \frac{du}{u} = \ln|u| + C$

$= \ln|e^x - e^{-x}| + C$

NAME: ______________________
STUDENT NO.: ______________
SECTION: __________________

[6] 9. Find the function $f(x)$ given that

$$f'(x) = \frac{3}{x} - 4$$

and the graph of $f(x)$ passes through the point $(1, 0)$.

FIND $f(x) = \int f'(x)\,dx$

$= 3\ln|x| - 4x + C$

NOW $f(1) = 0 \Rightarrow 1 = 3\ln 0 - 4\cdot 0 + C$

$C = -2$

$\Rightarrow$ SO $f(x) = 3\ln|x| - 4x - 2$

[5] 10. Find $\frac{dy}{dx}$ for the curve $4x^3 + 11xy^2 - 2y^3 = 0$.

$12x^2 + 11y^2 + 22xy\frac{dy}{dx} - 6y^2\frac{dy}{dx} = 0$

SOLVE: $\frac{dy}{dx} = \frac{12x^2 + 11y^2}{6y^2 - 22xy}$

NAME: ______________________
STUDENT NO.: ________________
SECTION: ____________________

11. Let $f(x) = \dfrac{x}{(x-1)^2}$

$$f'(x) = \frac{-(x+1)}{(x-1)^3} \qquad f''(x) = \frac{2(x+2)}{(x-1)^4}$$

For the function $f(x)$ find

[2] (a) The domain.

ALL $x \in \mathbb{R}$ EXCEPT $x = 1$.

[4] (b) Vertical and horizontal asymptotes (if any).

HORIZONTAL: $\lim_{x \to \infty} f(x) = 0$, SO $y = 0$ IS A H.A.

VERTICAL (CHECK $x=1$): $\lim_{x \to 1} f(x) = +\infty$ (FROM BOTH SIDES)

SO $x = 1$ IS A V.A.

NAME: ______________________
STUDENT NO.: ______________
SECTION: __________________

[5] (c) Intervals of increase and decrease, critical numbers and points, any relative maximas or minimas.

CRITICAL POINTS: $f'(x) = 0$ WHEN $x = -1$

$f'(x)$ DNE ~~WHEN~~ NO SUCH POINTS

TEST $x = -1$: $f''(-1) > 0 \Rightarrow (x, y) = (-1, -\frac{1}{4})$ LOCAL MIN

$\Rightarrow f(x)$ INCREASING ~~FROM~~ ON ~~x = -1 TO x~~ $-1 < x < 1$

DECREASING ~~FROM~~ ON $x < -1$ AND $x > 1$

[5] (d) Intervals of concave upward and concave downward, inflection points.

[100]

CANDIDATES: $f''(x) = 0$ WHEN $x = -2$

$f''(x)$ DNE NO SUCH POINTS

TEST $x = -2$:

$f''(-3) < 0$ -2 $f''(-1) > 0$

$\rightarrow (x, y) = (-2, -\frac{2}{9})$ IS I.P.

$\Rightarrow f(x)$ CONCAVE UP ON $-2 < x < 1$ AND $x > 1$

CONCAVE DOWN ON $x < -2$

(NOT ASKED, BUT HERE'S THE GRAPH: