UNIVERSITY OF REGINA
Department of Mathematics and Statistics

FIRST NATIONS UNIVERSITY
Science Department

Math 103-001,002,003,S01,S02

Final Exam, Fall 2009

Time: 3 hours Pages: 7

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Name:________________________

Student Number:________________

Section : ______________

Show all required work, explaining necessary steps. Use the back of each page if sufficient space is not available. Use scrap paper for rough work, and do not hand it in.

1. Find the indicated limit. If a limit does not exist, state so. *[6 marks]*

a) $\lim\limits_{x\to 2}\frac{x^2-6x+8}{4-x^2}$ $= \lim\limits_{x\to 2}\frac{(x-4)(x-2)}{(2-x)(2+x)}$

$= \lim\limits_{x\to 2}\frac{4-x}{2+x} = \frac{1}{2}$

b) $\lim\limits_{x\to+\infty}\frac{3+2x-5x^2}{3x^2-1}$ $= \lim\limits_{x\to\infty}\frac{-5+\frac{2}{x}+\frac{3}{x^2}}{3-\frac{1}{x^2}} = -\frac{5}{3}$

2. Use the limit definition of the derivative to find $f'(4)$ if $f(x)=\sqrt{x+5}$. *[8 marks]*

$f'(4) = \lim\limits_{h\to 0}\frac{\sqrt{(4+h)+5}-\sqrt{9}}{h}\cdot\frac{\sqrt{9+h}+\sqrt{9}}{\sqrt{9+h}+\sqrt{9}}$

$= \lim\limits_{h\to 0}\frac{9+h-9}{h\left[\sqrt{9+h}+3\right]}$

$= \lim\limits_{h\to 0}\frac{1}{\sqrt{9+h}+3} = \frac{1}{6}$

3. Find the indicated derivatives. You do not need to simplify your answers. *[15 marks]*

a) $f(x) = \dfrac{x^2+1}{x^3+2x+3}$ $f'(1) = ?$

$$f'(x) = \frac{2x(x^3+2x+3) - (x^2+1)(3x^2+2)}{(x^3+2x+3)^2}$$

$$f'(1) = \frac{12 - 10}{36} = \frac{1}{18}$$

b) $g(t) = e^{t^2+1}$ $g''(t) = ?$

$$g'(t) = 2t \cdot e^{t^2+1}$$

$$g''(t) = 2e^{t^2+1} + 4t^2 e^{t^2+1}$$

c) $f(x) = x^2 \ln(x)$ $f'(e) = ?$

$$f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$$

$$f'(e) = 2e \cdot 1 + e = 3e$$

d) $xy^3 = x^2 + y^2$ $\dfrac{dy}{dx} = ?$

$$y^3 + 3xy^2 \frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

SOLVE: $$\frac{dy}{dx} = \frac{2x - y^3}{3xy^2 - 2y}$$

e) $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ $f'(x) = ?$

$$f'(x) = \frac{\sqrt{x^2+1} - \frac{1}{2}x(x^2+1)^{-1/2}(2x)}{x^2+1}$$

4. Consider the graph of $f(x) = \dfrac{2}{1+x^2}$. Identify (i) the domain, (ii) all intercepts, (iii) all vertical and horizontal asymptotes, (iv) all relative and absolute extreme points and (v) all inflection points. Use your results to sketch a graph of $y = f(x)$. *[12 marks]*

(Note: you may use the fact that $f'(x) = \dfrac{-4x}{(1+x^2)^2}$ and $f''(x) = \dfrac{12x^2-4}{(1+x^2)^3}$)

(i) DOMAIN: ALL $x \in \mathbb{R}$ (DENOMINATOR NEVER ZERO!)

(ii) y-INT: $y = 2$

X-INT: NONE (NUMERATOR NEVER ZERO)

(iii) HORIZONTAL ASYMPTOTE: $\lim_{x\to\infty} f(x) = 0 \Rightarrow y = 0$

VERTICAL? NONE, NO HOLES IN DOMAIN.

(iv) CRITICAL POINTS: $f'(x) = 0$ WHEN $x = 0$

$f'(x)$ DNE $\rightarrow$ NO SUCH POINTS

TEST: $f''(0) < 0 \Rightarrow (x,y) = (0,2)$ LOCAL MAX (ALSO ABSOLUTE)

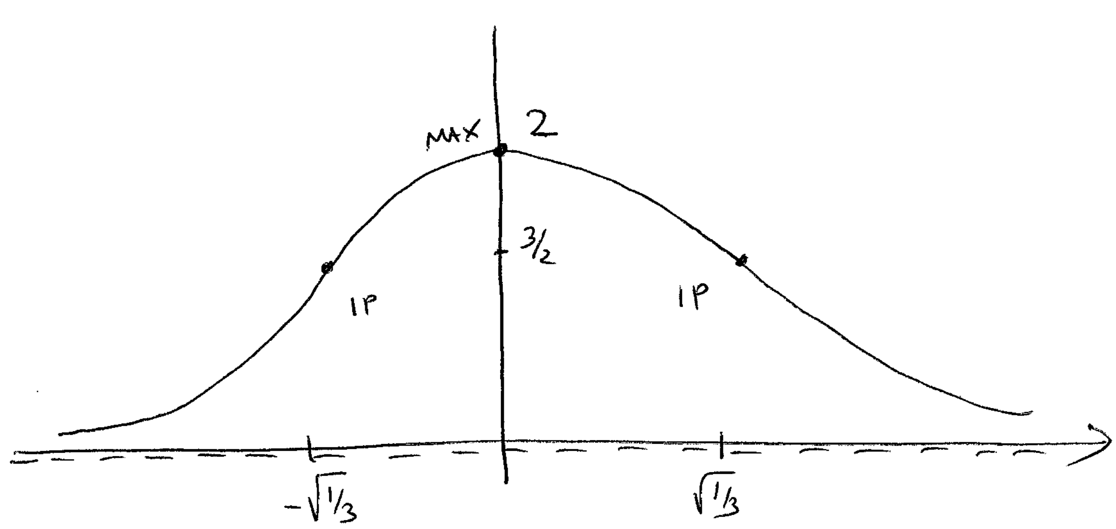
(v) INFLECTION POINTS: $f''(x) = 0$ WHEN $x = \pm\sqrt{\frac{1}{3}}$

$f''(x)$ DNE $\rightarrow$ NO SUCH POINTS

TEST:

$$\begin{array}{ccccc} \uparrow & -\sqrt{1/3} & \uparrow & +\sqrt{1/3} & \uparrow \\ f''>0 & \downarrow & f''<0 & \downarrow & f''>0 \\ & \text{I.P.} & & \text{I.P.} & \end{array}$$

SKETCH



5. Daily sales at a large multinational company are declining continuously at a yearly rate of 8%. If sales totaled 8.4 million today, and current trends continue, how long will it take for daily sales to drop below 4 million? Give your answer to the nearest month (e.g. "three years and five months"). *[7 marks]*

$N(t) = 8.4 e^{-0.08t}$

FIND t SUCH THAT $N(t) = 4$

SOLVE $4 = 8.4 e^{-0.08t}$

$\ln\left(\frac{40}{84}\right) = -0.08t \Rightarrow t = \frac{\ln\left(\frac{40}{84}\right)}{-0.08}$

≈ 9.27

(NINE YEARS AND 3 MONTHS)

6. The monthly output of a manufacturing plant is given by $Q = x^3 + 2xy^2 + y^3$ units, where x and y are the number of skilled and unskilled workers, respectively. Currently, there are x=120 skilled workers, however that number is falling at a rate of 10 workers per month. Similarly, unskilled labour consists of y=180 workers and is rising at a rate of 6 workers per month. *[10 marks]*

a) What is the current monthly output of this plant?

LET $x = 120$, $y = 180$

SO $Q = 15,336,000$ UNITS

b) What is the current rate of change of monthly output (in units per month)?

$$\frac{dQ}{dt} = 3x^2\frac{dx}{dt} + 2\frac{dx}{dt}y^2 + 4xy\frac{dy}{dt} + 3y^2\frac{dy}{dt}$$

(USE $\frac{dx}{dt} = -10$, $\frac{dy}{dt} = +6$)

PLUG IN, $\frac{dQ}{dt} = 21,600$,

OUTPUT IS RISING AT 21600 UNITS/MONTH.

7. An airline determines that it can sell 500 seats a day for the Toronto-Ottawa route at a price of \$640 each. Every \$10 price increase results in a sales drop of 5 seats. Use this information to determine the seat price that will maximize the airline's daily revenue.
[9 marks]

FIND PRICE $p = mx + b$

WITH SLOPE $m = \dfrac{10}{-5} = -2$

AND POINT $(x, p) = (500, 640)$

$\Rightarrow \quad 640 = (-2)(500) + b$

SOLVE FOR $b = 1640$

$\Rightarrow$ PRICE $p = -2x + 1640$

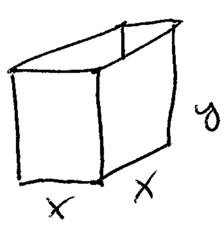
REVENUE $R(x) = -2x^2 + 1640x$

(MAXIMUM, SINCE PARABOLA)

NOW MAXIMIZE: $R'(x) = -4x + 1640 \Rightarrow$ CRITICAL POINT $x = \cancel{40}\ 410$

SO REVENUE IS MAXIMIZED IF $x = 410$, IE PRICE = \$820

8. An open rectangular storage container with square base is to be constructed. If the container must have a volume of 500 m^3, find the dimensions of the container that will have minimal surface area.
[9 marks]



VOLUME $V = x^2 y$

AREA $A = x^2 + 4xy$

USE $500 = x^2 y \Rightarrow y = \dfrac{500}{x^2}$

COMBINE: $A(x) = x^2 + \dfrac{2000}{x}$

NOW MINIMIZE: $A'(x) = 2x - 2000x^{-2}$

CRITICAL POINT: $A'(x) = 0 \Rightarrow x = 10$

$A'(x)$ DNE $\Rightarrow$ NO SUCH POINTS

TEST $x = 10$: $A''(x) = 2 + 4000x^{-3}$

$A''(10) > 0 \Rightarrow$ MINIMUM

ONLY ONE CRITICAL POINT, SO $x = 10$ ALSO GLOBAL MIN.

OPTIMAL DIMENSIONS ARE $10\text{m} \times 10\text{m} \times 5\text{m}$.

9. Integrate: *[16 marks]*

a) $\int \left(3e^x + x^3 + \sqrt[3]{x}\right)dx$

$= 3e^x + \frac{1}{4}x^4 + \frac{3}{4}x^{4/3} + C$

b) $\int \frac{x}{x^2+1}dx$

$u = x^2+1$
$du = 2x\,dx$

$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+1\right| + C$

c) $\int \frac{x+1+\sqrt{x}}{x^2}dx = \int \left(x^{-1} + x^{-2} + x^{-3/2}\right)dx$

$= \ln|x| - x^{-1} - 2x^{-1/2} + C$

d) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$.

$u = \sqrt{x}$

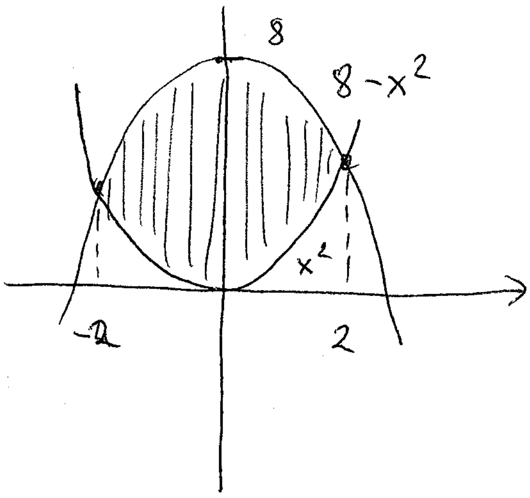
$\frac{du}{dx} = \frac{1}{2}x^{-1/2} \left(= \frac{1}{2\sqrt{x}}\right)$

$= 2\int e^u\,du = 2e^u + C$

$= 2e^{\sqrt{x}} + C$

10. Sketch the region bounded by the curves $y = x^2$ and $y = 8 - x^2$, and then find the area of the region. *[8 marks]*

PARABOLAS



NOTE INTERSECTION:

$$x^2 = 8 - x^2$$
$$2x^2 = 8$$
$$x = \pm 2$$

ALSO NOTE SYMMETRY!

AREA $$A = 2 \cdot \int_0^2 \left[(8 - x^2) - (x^2) \right] dx$$

$$= 2 \cdot \int_0^2 (8 - 2x^2)\, dx$$

$$= 2 \left[8x - \frac{2}{3} x^3 \right]_0^2$$

$$= 2 \left[16 - \frac{16}{3} \right] = \frac{64}{3} \text{ UNITS}^2$$