# Math 302.102 Fall 2010 Midterm #2 – Solutions

**1. (a)** The density function of $X$ satsifes

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} ax^{-a-1}, & x \ge 1, \\ 0, & x < 1. \end{cases}$$

**1. (b)** The mean of $X$ is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_1^{\infty} x \cdot ax^{-a-1}\,\mathrm{d}x = \int_1^{\infty} ax^{-a}\,\mathrm{d}x = \left.\frac{ax^{1-a}}{1-a}\right|_1^{\infty} = \frac{a}{a-1}.$$

**1. (c)** Since the second moment of $X$ is

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,\mathrm{d}x = \int_1^{\infty} x^2 \cdot ax^{-a-1}\,\mathrm{d}x = \int_1^{\infty} ax^{1-a}\,\mathrm{d}x = \left.\frac{ax^{2-a}}{2-a}\right|_1^{\infty} = \frac{a}{a-2},$$

we conclude that the variance of $X$ is

$$\mathrm{var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{a}{a-2} - \left(\frac{a}{a-1}\right)^2 = \frac{a}{(a-2)(a-1)^2}.$$

**1. (d)** Let $t > 1$ be fixed so that if $x \ge 0$, then $tx \ge x$. If $x > 1$, then by definition of conditional probability,

$$\mathbf{P}\{X \ge tx \,|\, X \ge x\} = \frac{\mathbf{P}\{X \ge tx \cap X \ge x\}}{\mathbf{P}\{X \ge x\}} = \frac{\mathbf{P}\{X \ge tx\}}{\mathbf{P}\{X \ge x\}} = \frac{1 - F_X(tx)}{1 - F_X(x)} = \frac{(tx)^{-a}}{x^{-a}} = t^{-a}.$$

If $0 \le x \le 1$, then $\mathbf{P}\{X \ge x\} = 1$ and so $\mathbf{P}\{X \ge tx \,|\, X \ge x\} = 1 - F_X(tx)$. Finally, if $x < 0$, then $tx < x$ for any $t > 1$ so $\mathbf{P}\{X \ge tx \,|\, X \ge x\} = 1$.

**1. (e)** If $Y = X^a$, then for $y > 1$, the distribution function of $Y$ is

$$F_Y(y) = \mathbf{P}\{Y \le y\} = \mathbf{P}\{X^a \le y\} = \mathbf{P}\left\{X \le y^{1/a}\right\} = \int_1^{y^{1/a}} f_X(x)\,\mathrm{d}x.$$

Thus, by the fundamental theorem of calculus, if $y > 1$, then

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = f_X(y^{1/a}) \cdot \frac{\mathrm{d}}{\mathrm{d}y} y^{1/a} = a\left(y^{1/a}\right)^{-a-1} \cdot \frac{1}{a} y^{1/a-1} = ay^{-1-1/a} \cdot \frac{1}{a} y^{1/a-1} = y^{-2}.$$

**2. (a)** We find

$$\begin{aligned}\mathbf{P}\left\{-2 < X \le \frac{1}{2}\right\} = \int_{-2}^{1/2} f_X(x)\,\mathrm{d}x = \int_{-2}^{-1} f_X(x)\,\mathrm{d}x + \int_{-1}^{1/2} f_X(x)\,\mathrm{d}x = 0 + \int_{-1}^{1/2} \frac{1}{2}(x+1)\,\mathrm{d}x \\ = \left.\frac{1}{4}(x+1)^2\right|_{-1}^{1/2} \\ = \frac{9}{16}.\end{aligned}$$

**2. (b)** If $Y = |X|$, then for $0 \le y \le 1$, the distribution function of $Y$ is

$$\begin{aligned} F_Y(y) = \mathbf{P}\{Y \le y\} = \mathbf{P}\{|X| \le y\} = \mathbf{P}\{-y \le X \le y\} &= \int_{-y}^{y} f_X(x)\,\mathrm{d}x \\ &= \int_0^y f_X(x)\,\mathrm{d}x + \int_{-y}^{0} f_X(x)\,\mathrm{d}x \\ &= \int_0^y f_X(x)\,\mathrm{d}x - \int_0^{-y} f_X(x)\,\mathrm{d}x. \end{aligned}$$

Thus, by the fundamental theorem of calculus, if $0 \le y \le 1$, then

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) = f_X(y)\cdot\frac{\mathrm{d}}{\mathrm{d}y}y - f_X(-y)\frac{\mathrm{d}}{\mathrm{d}y}(-y) = f_X(y) + f_X(-y) = \frac{1}{2}(y+1) + \frac{1}{2}(-y+1) = 1.$$

Note that $f_Y(y) = 1$ for $0 \le y \le 1$ so that $Y \sim \mathrm{Unif}(0,1)$.

**3.** We want to compute $\mathbb{E}(Y)$ where $Y$ denotes the amount of time until Alice's computer system stops operating. However, $Y = \min\{X_1, X_2, X_3\}$ where $X_j$ denotes the lifetime of the $j$th circuit board. Thus, if $y \ge 0$, then the distribution function of $Y$ satisfies

$$\begin{aligned} F_Y(y) = \mathbf{P}\{Y \le y\} = 1 - \mathbf{P}\{Y > y\} &= 1 - \mathbf{P}\{\min\{X_1, X_2, X_3\} > y\} \\ &= 1 - \mathbf{P}\{X_1 > y, X_2 > y, X_3 > y\} \\ &= 1 - \mathbf{P}\{X_1 > y\}\,\mathbf{P}\{X_2 > y\}\,\mathbf{P}\{X_3 > y\} \\ &= 1 - [\mathbf{P}\{X > y\}]^3. \end{aligned}$$

Now,

$$\mathbf{P}\{X > y\} = \int_y^\infty f_X(x)\,\mathrm{d}x = \int_y^\infty \frac{2}{(x+1)^3}\,\mathrm{d}x = -\frac{1}{(x+1)^2}\bigg|_y^\infty = \frac{1}{(y+1)^2}$$

so that

$$F_Y(y) = 1 - \left(\frac{1}{(y+1)^2}\right)^3 = 1 - \frac{1}{(y+1)^6}.$$

Thus, if $y \ge 0$, then the density function of $Y$ is

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) = \frac{6}{(y+1)^7}.$$

Finally, we conclude that

$$\begin{aligned} \mathbb{E}(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,\mathrm{d}y = \int_0^\infty y \cdot \frac{6}{(y+1)^7}\,\mathrm{d}y = \int_1^\infty \frac{6(u-1)}{u^7}\,\mathrm{d}u &= \int_1^\infty (6u^{-6} - 6u^{-7})\,\mathrm{d}u \\ &= \left[\frac{1}{u^6} - \frac{6}{5u^5}\right]_1^\infty \\ &= \frac{6}{5} - 1 \\ &= \frac{1}{5}. \end{aligned}$$

**4.** Notice that Yolanda will leave before Xavier arrives if and only if Xavier arrives more than one hour after Yolanda arrives. Symbolically, this corresponds to the event $\{X > Y + 1\}$. Hence, by the law of total probability (since $f_Y(y) = 1$ for $0 \le y \le 1$ it is easiest to condition on the value of $Y$),

$$\begin{aligned}
\mathbf{P}\{X > Y + 1\} = \int_{-\infty}^{\infty} \mathbf{P}\{X > y + 1\} f_Y(y)\,\mathrm{d}y &= \int_{-\infty}^{\infty} \left[\int_{y+1}^{\infty} f_X(x)\,\mathrm{d}x\right] f_Y(y)\,\mathrm{d}y \\
&= \int_0^1 \int_{y+1}^{\infty} e^{-x}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 e^{-y-1}\,\mathrm{d}y \\
&= -e^{-y-1}\Big|_0^1 \\
&= e^{-1} - e^{-2}.
\end{aligned}$$