Math 302.102 Fall 2010 Midterm #1 (version 2) – Solutions

1. **(a)** We must choose c so that

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^2 cx^3\,\mathrm{d}x.$$

Since

$$\int_0^2 x^3\,\mathrm{d}x = \frac{x^4}{4}\bigg|_0^2 = \frac{2^4}{4} - \frac{0^4}{4} = 4$$

we conclude that $c = 1/4$.

(b) By definition, $F(x) = \mathbf{P}\{X \le x\}$. Note that if $x \le 0$, then $F(x) = 0$, and if $x \ge 2$, then $F(x) = 1$. If $0 < x < 2$, then

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x \frac{t^3}{4}\,\mathrm{d}t = \frac{t^4}{16}\bigg|_0^x = \frac{x^4}{16}.$$

Hence,

$$F(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \dfrac{x^4}{16}, & \text{if } 0 < x < 2, \\ 1, & \text{if } x \ge 2. \end{cases}$$

(c) We find

$$\mathbf{P}\{X > 1\} = 1 - \mathbf{P}\{X \le 1\} = 1 - F(1) = 1 - \frac{1^4}{16} = \frac{15}{16}.$$

Alternatively,

$$\mathbf{P}\{X > 1\} = \int_1^{\infty} f(x)\,\mathrm{d}x = \int_1^2 \frac{x^3}{4}\,\mathrm{d}x = \frac{x^4}{16}\bigg|_1^2 = \frac{2^4}{16} - \frac{1^4}{16} = \frac{15}{16}.$$

2. Observe that

$$\begin{aligned} &\mathbf{P}\{\text{mechanical system operates at least 3 days}\} \\ &\qquad = 1 - \mathbf{P}\{\text{mechanical system does not operate at least 3 days}\} \\ &\qquad = 1 - \mathbf{P}\{\text{both components fail within 3 days}\} \\ &\qquad = 1 - \mathbf{P}\{\text{component 1 fails within 3 days}\}\,\mathbf{P}\{\text{component 2 fails within 3 days}\} \end{aligned}$$

where the last equality follows since the two components are assumed to perform independently. Since component i has an exponential distribution with parameter λ_i, we find

$$\mathbf{P}\{\text{component } i \text{ fails within 3 days}\} = \int_0^3 \lambda_i e^{-\lambda_i x}\,\mathrm{d}x = -e^{-\lambda_i x}\bigg|_0^3 = 1 - e^{-3\lambda_i}.$$

Since $\lambda_1 = 1$ and $\lambda_2 = 5$, the required probability is

$$\begin{aligned} \mathbf{P}\{\text{mechanical system operates at least 3 days}\} &= 1 - [1 - e^{-3\lambda_1}][1 - e^{-3\lambda_2}] \\ &= 1 - [1 - e^{-3}][1 - e^{-15}]. \end{aligned}$$

3. The key to answering this question is to recall that if A and B are any events, then

$$\mathbf{P}\{A \cup B\} = \mathbf{P}\{A\} + \mathbf{P}\{B\} - \mathbf{P}\{A \cap B\}.$$

(a) If A and B are disjoint events, then $\mathbf{P}\{A \cap B\} = 0$. Thus,

$$0.8 = 0.4 + \mathbf{P}\{B\} + 0$$

implying that $\mathbf{P}\{B\} = 0.8 - 0.4 = 0.4$.

(b) If A and B are independent events, then $\mathbf{P}\{A \cap B\} = \mathbf{P}\{A\}\mathbf{P}\{B\} = 0.4\mathbf{P}\{B\}$. Thus,

$$0.8 = 0.4 + \mathbf{P}\{B\} - 0.4\mathbf{P}\{B\}$$

implying that

$$\mathbf{P}\{B\} = \frac{0.8 - 0.4}{1 - 0.4} = \frac{0.4}{0.6} = \frac{2}{3}.$$

(c) By definition, $\mathbf{P}\{A \cap B\} = \mathbf{P}\{B|A\}\mathbf{P}\{A\} = (0.3)(0.4) = 0.12$. Thus,

$$0.8 = 0.4 + \mathbf{P}\{B\} - 0.12$$

implying that $\mathbf{P}\{B\} = 0.8 - 0.4 + 0.12 = 0.52$.

4. Let W_j, R_j, B_j denote the event that a white ball, red ball, black ball, respectively, was selected on draw j.

(a) Using the law of total probability we find

$$\begin{aligned}
\mathbf{P}\{R_2\} &= \mathbf{P}\{R_2 \,|\, R_1\}\mathbf{P}\{R_1\} + \mathbf{P}\{R_2 \,|\, W_1\}\mathbf{P}\{W_1\} + \mathbf{P}\{R_2 \,|\, B_1\}\mathbf{P}\{B_1\} \\
&= \frac{8}{18} \cdot \frac{5}{15} + \frac{5}{18} \cdot \frac{4}{15} + \frac{5}{18} \cdot \frac{6}{15} \\
&= \frac{5}{15}\left[\frac{8}{18} + \frac{4}{18} + \frac{6}{18}\right] \\
&= \frac{5}{15}.
\end{aligned}$$

Notice that the probability that the second ball is red is the *same* as the probability that the first ball is red; that is, $\mathbf{P}\{R_1\} = \mathbf{P}\{R_2\} = 5/15$. However, the events R_1 and R_2 are *not* independent. This is because

$$\mathbf{P}\{R_1 \cap R_2\} = \mathbf{P}\{R_2 \,|\, R_1\}\mathbf{P}\{R_1\} = \frac{8}{18} \cdot \frac{5}{15}$$

which does not equal $\mathbf{P}\{R_1\}\mathbf{P}\{R_2\}$.

4. (b) Using Bayes' rule we find

$$\mathbf{P}\{B_1 \,|\, R_2\} = \frac{\mathbf{P}\{R_2 \,|\, B_1\}\mathbf{P}\{B_1\}}{\mathbf{P}\{R_1\}} = \frac{\frac{5}{18} \cdot \frac{6}{15}}{\frac{8}{18} \cdot \frac{5}{15} + \frac{5}{18} \cdot \frac{4}{15} + \frac{5}{18} \cdot \frac{6}{15}} = \frac{30}{40 + 20 + 30} = \frac{1}{3}.$$

5. Let X denote the number of passengers who do not show up so that X has a binomial distribution with $n = 122$ trials and probability of success $p = 0.06$ on each trial. Note that we are defining "success" as "passenger does not show up."

Observe that there will be a seat for every passenger if and only if at least 2 passengers do not show up. Thus,

$$\begin{aligned}
&\mathbf{P}\{\text{there will be a seat for every passenger}\} \\
&\quad = \mathbf{P}\{X \geq 2\} \\
&\quad = 1 - \mathbf{P}\{X < 2\} \\
&\quad = 1 - \mathbf{P}\{X = 0\} - \mathbf{P}\{X = 1\} \\
&\quad = 1 - \binom{122}{0}(0.06)^0(0.94)^{122} - \binom{122}{1}(0.06)^1(0.94)^{121} \\
&\quad = 1 - (0.94)^{122} - 122(0.06)(0.94)^{121}.
\end{aligned}$$

Although it was not required, this expression for the probability can be computed. It is approximately 0.995371.