Math 302.102 Fall 2010
Class Exercises from October 20, 2010

Remark. There was some discussion last class regarding the notation D_j that was defined on *The Birthday Problem* handout. The error was that instead of having $D_j \subset D_i$, I originally wrote $D_i \subset D_j$. Thus, when I tried to correct it using unions, I was not correct. This is the correct statement.

Also notice that if $i < j$, then $D_j \subset D_i$. This says that if the first j birthdays are different and i is less than j, then the first i birthdays must certainly be different. Hence,

$$D_j = D_1 \cap D_2 \cap \cdots \cap D_j.$$

Example. Suppose that Alice and Bob each roll a die. Alice wins if her roll is strictly larger than Bob's roll. What is the probability that Alice wins?

Solution. Let X denote the result of Alice's roll, and let Y denote the result of Bob's roll. Note that Alice wins if and only if $X > Y$. Hence, we must compute $\mathbf{P}\{X > Y\}$. One way to do this is to write out a table of all 36 possibilities for their rolls, and to then check which produce a win for Alice. In this way, we find

$$\mathbf{P}\{X > Y\} = \frac{15}{36}.$$

However, another solution is to use the law of total probability and condition on the resulting value of Bob's roll. That is,

$$\begin{aligned}\mathbf{P}\{X > Y\} &= \mathbf{P}\{X > Y \,|\, Y = 1\}\,\mathbf{P}\{Y = 1\} + \cdots + \mathbf{P}\{X > Y \,|\, Y = 6\}\,\mathbf{P}\{Y = 6\} \\ &= \sum_{y=1}^{6} \mathbf{P}\{X > Y \,|\, Y = y\}\,\mathbf{P}\{Y = y\}.\end{aligned}$$

Since each of the six outcomes is equally likely, we find

$$\mathbf{P}\{Y = y\} = \frac{1}{6}$$

for $y = 1, 2, \ldots, 6$. Furthermore, we find

$$\mathbf{P}\{X > Y \,|\, Y = y\} = \frac{6-y}{6}.$$

Thus,

$$\mathbf{P}\{X > Y\} = \sum_{y=1}^{6} \mathbf{P}\{X > Y \,|\, Y = y\}\,\mathbf{P}\{Y = y\} = \frac{1}{6}\sum_{y=1}^{6} \frac{6-y}{6} = \frac{1}{36}(5+4+3+2+1) = \frac{15}{36}.$$

Example. Suppose that Alice gets to roll two dice, but Bob only gets to roll one. Alice wins if the largest of her two rolls is strictly larger than Bob's roll. What is the probability that Alice wins?

Solution. Suppose that we denote the results of Alice's two rolls by X_1 and X_2, respectively. Let Y denote the result of Bob's roll. Note that Alice wins if and only if $\max\{X_1, X_2\} > Y$. Therefore, let $X = \max\{X_1, X_2\}$ so that we are interested in computing $\mathbf{P}\{X > Y\}$. By conditioning on the resulting value of Bob's roll, we find

$$\mathbf{P}\{X > Y\} = \sum_{y=1}^{6} \mathbf{P}\{X > Y \mid Y = y\} \mathbf{P}\{Y = y\} = \frac{1}{6} \sum_{y=1}^{6} \mathbf{P}\{X > Y \mid Y = y\}.$$

However, we still need to compute $\mathbf{P}\{X > Y \mid Y = y\} = \mathbf{P}\{\max\{X_1, X_2\} > Y \mid Y = y\}$. Here is the trick. Observe that if we know an upper bound on the maximum of two numbers, then we know an upper bound on each of the numbers. That is,

$$\{\max\{X_1, X_2\} \le a\} \text{ if and only if } \{X_1 \le a \text{ and } X_2 \le a\}.$$

However, since X_1 and X_2 are independent, we find

$$\mathbf{P}\{X_1 \le a \text{ and } X_2 \le a\} = \mathbf{P}\{X_1 \le a\} \mathbf{P}\{X_2 \le a\}.$$

Thus, we find

$$\begin{aligned}
\mathbf{P}\{X > Y \mid Y = y\} &= \mathbf{P}\{\max\{X_1, X_2\} > Y \mid Y = y\} \\
&= 1 - \mathbf{P}\{\max\{X_1, X_2\} \le Y \mid Y = y\} \\
&= 1 - \mathbf{P}\{X_1 \le Y \mid Y = y\} \mathbf{P}\{X_2 \le Y \mid Y = y\} \\
&= 1 - \left(\frac{y}{6}\right)^2.
\end{aligned}$$

Note that if we are given the resulting value of Bob's roll, say $Y = y$, then there are precisely y values of X_1 that are less than or equal to y, and similarly for X_2. That is,

$$\mathbf{P}\{X_1 \le Y \mid Y = y\} = \mathbf{P}\{X_2 \le Y \mid Y = y\} = \frac{y}{6}$$

which explains the last equality above. Combining everything gives

$$\mathbf{P}\{X > Y\} = \frac{1}{6} \sum_{y=1}^{6} \mathbf{P}\{X > Y \mid Y = y\} = \frac{1}{6} \sum_{y=1}^{6} \left[1 - \left(\frac{y}{6}\right)^2\right] = \frac{125}{216}.$$

Example. Suppose that Alice and Bob each have a lightbulb. It is known that the lightbulbs operate independently and that the lifetime of each lightbulb is exponentially distributed with parameter $\lambda = 2$. Alice wins if her lightbulb outlasts Bob's lightbulb. What is the probability that Alice wins?