Math 302.102 Fall 2010
Summary of Lecture from September 22, 2010

The purpose of today's class was to derive two of the most useful formulas in probability, namely the *Law of Total Probability* and *Bayes' Rule*. With a bit of guidance, it is not so difficult to derive these results.

Suppose that A and B are events with $\mathbf{P}\{A\} > 0$ and $\mathbf{P}\{B\} > 0$. From a Venn diagram, we see that A can be decomposed into two pieces, namely (i) that part of A which is also in B, and (ii) that part of A which is not in B. In other words, $A = (A \cap B) \cup (A \cap B^c)$ expresses A as a disjoint union. Using the property that says the probability of a disjoint union is the sum of the probabilities, we conclude

$$\mathbf{P}\{A\} = \mathbf{P}\{(A \cap B) \cup (A \cap B^c)\} = \mathbf{P}\{A \cap B\} + \mathbf{P}\{A \cap B^c\}. \qquad (*)$$

Recall that the definition of the conditional probability of A given B is

$$\mathbf{P}\{A \,|\, B\} = \frac{\mathbf{P}\{A \cap B\}}{\mathbf{P}\{B\}}.$$

Solving for $\mathbf{P}\{A \cap B\}$ gives

$$\mathbf{P}\{A \cap B\} = \mathbf{P}\{A \,|\, B\}\,\mathbf{P}\{B\}. \qquad (\dagger)$$

Similarly, we find

$$\mathbf{P}\{A \cap B^c\} = \mathbf{P}\{A \,|\, B^c\}\,\mathbf{P}\{B^c\}.$$

Substituting the previous two expressions into $(*)$ gives the **Law of Total Probability**.

$$\boxed{\mathbf{P}\{A\} = \mathbf{P}\{A \,|\, B\}\,\mathbf{P}\{B\} + \mathbf{P}\{A \,|\, B^c\}\,\mathbf{P}\{B^c\}}$$

However, we can also consider the conditional probability of B given A which is

$$\mathbf{P}\{B \,|\, A\} = \frac{\mathbf{P}\{B \cap A\}}{\mathbf{P}\{A\}}$$

so that solving for $\mathbf{P}\{B \cap A\}$ gives

$$\mathbf{P}\{B \cap A\} = \mathbf{P}\{B \,|\, A\}\,\mathbf{P}\{A\}. \qquad (\ddagger)$$

But, of course, the events that $A \cap B$ and $B \cap A$ are the same! This means that $(\dagger)$ and $(\ddagger)$ give two different, but equal, expressions for $\mathbf{P}\{A \cap B\}$. Equating them gives

$$\mathbf{P}\{A \,|\, B\}\,\mathbf{P}\{B\} = \mathbf{P}\{B \,|\, A\}\,\mathbf{P}\{A\}.$$

Finally, we can solve this for $\mathbf{P}\{B \,|\, A\}$. This leads to the first version of **Bayes' Rule**.

$$\boxed{\mathbf{P}\{B \,|\, A\} = \frac{\mathbf{P}\{A \,|\, B\}\,\mathbf{P}\{B\}}{\mathbf{P}\{A\}}.}$$

Finally, if we substitute the expression for $\mathbf{P}\{A\}$ from the Law of Total Probability into the first version of Bayes' Rule, then we arrive at another version of **Bayes' Rule**.

$$\boxed{\mathbf{P}\{B \,|\, A\} = \frac{\mathbf{P}\{A \,|\, B\}\,\mathbf{P}\{B\}}{\mathbf{P}\{A \,|\, B\}\,\mathbf{P}\{B\} + \mathbf{P}\{A \,|\, B^c\}\,\mathbf{P}\{B^c\}}.}$$

The importance of Bayes' Rule is that it tells us that if we know the conditional probability of A given B, then we can determine the conditional probability of B given A.

Example. Suppose that patients are randomly assigned to one of two treatment regimes. After completing a course of treatment, the investigator examines each patient and assesses whether or not the patient has shown marked improvement. Assume that 60% of patients are assigned to treatment 1 and the remaining 40% are assigned to treatment 2. Assume further that 70% of those receiving treatment 1 show marked improvement and 65% of those receiving treatment 2 show marked improvement. Suppose that one patient is selected at random.

- **(a)** Determine the probability the patient shows marked improvement.
- **(b)** If the patient shows marked improvement, determine the probability that the patient received treatment 1.

Solution. Let T_1 and T_2 denote the events that the patient received treatments 1 and 2, respectively. Let I be the event that the patient shows marked improvement. In terms of probabilities, the information given in the problem is that

$$\mathbf{P}\{T_1\} = 0.6, \quad \mathbf{P}\{T_2\} = 0.4,$$

$$\mathbf{P}\{I \,|\, T_1\} = 0.70, \quad \mathbf{P}\{I^c \,|\, T_1\} = 0.30, \quad \mathbf{P}\{I \,|\, T_2\} = 0.65, \quad \mathbf{P}\{I^c \,|\, T_2\} = 0.35.$$

(a) The Law of Total Probability implies that

$$\begin{aligned}\mathbf{P}\{I\} = \mathbf{P}\{I \cap T_1\} + \mathbf{P}\{I \cap T_2\} &= \mathbf{P}\{I \,|\, T_1\}\,\mathbf{P}\{T_1\} + \mathbf{P}\{I \,|\, T_2\}\,\mathbf{P}\{T_2\} \\ &= (0.70)(0.60) + (0.65)(0.60) \\ &= 0.68.\end{aligned}$$

(b) The first version of Bayes' Rule implies that

$$\mathbf{P}\{T_1 \,|\, I\} = \frac{\mathbf{P}\{I \,|\, T_1\}\,\mathbf{P}\{T_1\}}{\mathbf{P}\{I\}} = \frac{(0.70)(0.60)}{0.68} = \frac{42}{68}.$$