Math 302.102 Fall 2010
Solution to First Problem from the Binomial Distribution Handout

Problem 1. Suppose that a fair coin was tossed 20 times and that there were 12 heads observed. (You may assume that the results of subsequent tosses were independent.)

(a) What is the probability that the first toss showed heads?

(b) What is the probability that the first two tosses showed heads?

(c) What is the probability that at least two of the first five tosses landed heads?

Solution. Let X denote the number of heads observed in 20 tosses of the coin.

(a) Using Bayes' rule, we find

$$\mathbf{P}\{\text{1st toss showed heads} \,|\, X = 12\} = \frac{\mathbf{P}\{X = 12 \,|\, \text{1st toss showed heads}\}\,\mathbf{P}\{\text{1st toss showed heads}\}}{\mathbf{P}\{X = 12\}}.$$

Clearly,

$$\mathbf{P}\{X = 12\} = \binom{20}{12}(1/2)^{12}(1/2)^{8}$$

and

$$\mathbf{P}\{\text{1st toss showed heads}\} = \frac{1}{2}.$$

Notice that if the 1st toss showed heads, then the only way for $X = 12$ is if there are 11 heads in the remaining 19 tosses. Thus,

$$\mathbf{P}\{X = 12 \,|\, \text{1st toss showed heads}\} = \binom{19}{11}(1/2)^{11}(1/2)^{9}.$$

Combining everything gives

$$\mathbf{P}\{\text{1st toss showed heads} \,|\, X = 12\} = \frac{\binom{19}{11}(1/2)^{11}(1/2)^{9}\cdot(1/2)}{\binom{20}{12}(1/2)^{12}(1/2)^{8}} = \frac{\binom{19}{11}}{\binom{20}{12}} = \frac{12}{20}.$$

(b) Using Bayes' rule, we find as in **(a)** that

$$\mathbf{P}\{\text{1st two tosses showed heads} \,|\, X = 12\} = \frac{\mathbf{P}\{X = 12 \,|\, \text{1st two tosses showed heads}\}\,\mathbf{P}\{\text{1st two tosses showed heads}\}}{\mathbf{P}\{X = 12\}}.$$

If the 1st two tosses showed heads, then the only way for $X = 12$ is if there are 10 heads in the remaining 18 tosses. Thus,

$$\mathbf{P}\{X = 12 \,|\, \text{1st two tosses showed heads}\} = \binom{18}{10}(1/2)^{10}(1/2)^{8}.$$

Thus,

$$\mathbf{P}\{\text{1st two tosses showed heads} \,|\, X = 12\} = \frac{\binom{18}{10}(1/2)^{10}(1/2)^8 \cdot (1/2)^2}{\binom{20}{12}(1/2)^{12}(1/2)^8} = \frac{\binom{18}{10}}{\binom{20}{12}} = \frac{12}{20} \cdot \frac{11}{19}.$$

(c) Observe that the event {at least 2 of the first 5 tosses landed heads} can be written as the union of the events

$$\{\text{exactly 2 of the first 5 landed heads}\} \cup \cdots \cup \{\text{exactly 5 of the first 5 landed heads}\}$$

Thus, using Bayes' rule, we find

$$\begin{aligned}&\mathbf{P}\{\text{exactly 2 of the first 5 landed heads} \,|\, X = 12\} \\ &\quad = \frac{\mathbf{P}\{X = 12 \,|\, \text{exactly 2 of the first 5 landed heads}\}\,\mathbf{P}\{\text{exactly 2 of the first 5 landed heads}\}}{\mathbf{P}\{X = 12\}}.\end{aligned}$$

Notice that

$$\mathbf{P}\{\text{exactly 2 of the first 5 landed heads}\} = \binom{5}{2}(1/2)^2(1/2)^3.$$

Furthermore, if exactly 2 of the first 5 tosses showed heads, then the only way for $X = 12$ is if there are 10 heads in the remaining 15 tosses; that is,

$$\mathbf{P}\{X = 12 \,|\, \text{exactly 2 of the first 5 landed heads}\} = \binom{15}{10}(1/2)^{10}(1/2)^5.$$

Combined, we conclude

$$\mathbf{P}\{\text{exactly 2 of the first 5 landed heads} \,|\, X = 12\} = \frac{\binom{15}{10}(1/2)^{10}(1/2)^5 \cdot \binom{5}{2}(1/2)^2(1/2)^3}{\binom{20}{12}(1/2)^{12}(1/2)^8} = \frac{\binom{15}{10} \cdot \binom{5}{2}}{\binom{20}{12}}.$$

Similarly, we can find the probability that exactly k of the first 5 tosses landed heads given that $X = 12$. Piecing everything back together gives

$$\mathbf{P}\{\text{at least 2 of the first 5 landed heads} \,|\, X = 12\} = \frac{\binom{15}{10} \cdot \binom{5}{2}}{\binom{20}{12}} + \frac{\binom{15}{9} \cdot \binom{5}{3}}{\binom{20}{12}} + \frac{\binom{15}{8} \cdot \binom{5}{4}}{\binom{20}{12}} + \frac{\binom{15}{7} \cdot \binom{5}{5}}{\binom{20}{12}}.$$