Math 302.102 Fall 2010
Solutions to Prerequisite Review Exercises

Single-Variable Calculus

1. Using a substitution with $u = 2x$ gives

$$\int_0^\infty e^{-2x}\,\mathrm{d}x = -\frac{1}{2}e^{-2x}\bigg|_0^\infty = \frac{1}{2}.$$

2. Using parts with $u = x$ and $\mathrm{d}v = e^{-2x}\,\mathrm{d}x$ gives

$$\int_0^\infty xe^{-2x}\,\mathrm{d}x = -\frac{1}{2}xe^{-2x}\bigg|_0^\infty + \frac{1}{2}\int_0^\infty e^{-2x}\,\mathrm{d}x = 0 + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

3. Using parts with $u = x^2$ and $\mathrm{d}v = e^{-2x}\,\mathrm{d}x$ gives

$$\int_0^\infty x^2e^{-2x}\,\mathrm{d}x = -\frac{1}{2}x^2e^{-2x}\bigg|_0^\infty + \int_0^\infty xe^{-2x}\,\mathrm{d}x = 0 + \frac{1}{4} = \frac{1}{4}.$$

4. Using parts with $u = x^3$ and $\mathrm{d}v = e^{-2x}\,\mathrm{d}x$ gives

$$\int_0^\infty x^3e^{-2x}\,\mathrm{d}x = -\frac{1}{2}x^3e^{-2x}\bigg|_0^\infty + \frac{3}{2}\int_0^\infty x^2e^{-2x}\,\mathrm{d}x = 0 + \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{8}.$$

5. Using a substitution with $u = x^{1/3}$ gives

$$\int_0^\infty x^{-2/3}e^{-x^{1/3}}\,\mathrm{d}x = \int_0^\infty 3e^{-u}\,\mathrm{d}u = -3e^{-u}\bigg|_0^\infty = 3.$$

6. Using a substitution with $u = x^{1/a}$ gives

$$\int_0^\infty x^{1/a-1}e^{-x^{1/a}}\,\mathrm{d}x = \int_0^\infty ae^{-u}\,\mathrm{d}u = -ae^{-u}\bigg|_0^\infty = a.$$

7. Using a substitution with $u = x^{1/3}$ gives

$$\int_0^\infty x^{1/3}e^{-2x^{1/3}}\,\mathrm{d}x = \int_0^\infty 3u^3e^{-2u}\,\mathrm{d}u = 3\cdot\frac{3}{8} = \frac{9}{8}.$$

8. Using a substitution with $u = x^2$ gives

$$\int_0^\infty xe^{-x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^\infty e^{-u}\,\mathrm{d}u = -\frac{1}{2}e^{-u}\bigg|_0^\infty = \frac{1}{2}.$$

9. Using a substitution with $u = ax^2$ gives

$$\int_0^\infty xe^{-ax^2}\,\mathrm{d}x = \frac{1}{2a}\int_0^\infty e^{-u}\,\mathrm{d}u = -\frac{1}{2a}e^{-u}\bigg|_0^\infty = \frac{1}{2a}.$$

10. We will discuss this exercise in detail in class later this term.

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12. Using a substitution with $u = 1 - x$ gives

$$\int_0^1 x(1-x)^3 \, \mathrm{d}x = -\int_1^0 (1-u)u^3 \, \mathrm{d}u = \int_0^1 u^3(1-u) \, \mathrm{d}u = \int_0^1 u^3 - u^4 \, \mathrm{d}u = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}.$$

13. Using a substitution with $u = 1 - x$ gives

$$\int_0^1 x^2(1-x)^3 \, \mathrm{d}x = -\int_1^0 (1-u)^2u^3 \, \mathrm{d}u = \int_0^1 u^3(1-u)^2 \, \mathrm{d}u = \int_0^1 u^3 - 2u^4 + u^5 \, \mathrm{d}u = \frac{1}{4} - \frac{2}{5} + \frac{1}{6} = \frac{1}{60}.$$

14. Recognizing the antiderivative directly gives

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \, \mathrm{d}x = \arctan(x) \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

15. Using a substitution with $u = x^2 + 1$ gives

$$\int_0^{\infty} \frac{x}{x^2+1} \, \mathrm{d}x = \int_1^{\infty} \frac{1}{2u} \, \mathrm{d}u = \frac{1}{2} \log|u| \Big|_1^{\infty} = \infty.$$

Thus, the value of this integral does not exist as a real number.

16. By writing

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1} \, \mathrm{d}x = \int_0^{\infty} \frac{x}{x^2+1} \, \mathrm{d}x + \int_{-\infty}^{0} \frac{x}{x^2+1} \, \mathrm{d}x = \int_0^{\infty} \frac{x}{x^2+1} \, \mathrm{d}x - \int_0^{\infty} \frac{x}{x^2+1} \, \mathrm{d}x = \infty - \infty,$$

we see that the value of this integral does not exist. (Recall that $\infty - \infty$ is a so-called *indeterminant form*). Note that the integrand is an odd function, and so we might be tempted to say that the integral of an odd function over a symmetric interval is 0. While this fact is true for symmetric *finite* intervals $(-a, a)$, we need to be careful when the symmetric interval is $(-\infty, \infty)$. With this particular integral there is an infinite area above the axis to the right of 0 as well as an infinite area below the axis to the left of 0. Again, we might be tempted to say that these areas are *equal* and so they cancel out giving a value of 0 to the integral. But ∞ is not a real number and cannot be manipulated like that. We cannot say that $\infty - \infty = 0$. Thus, we must conclude that the value of this integral does not exist.

17. Recognizing the antiderivative directly gives

$$\int_a^{\infty} \frac{1}{x^3} \, \mathrm{d}x = -\frac{1}{2}x^{-2} \Big|_a^{\infty} = \frac{1}{2a^2}.$$

18. Recognizing the antiderivative directly gives

$$\int_a^{\infty} \frac{1}{x^b} \, \mathrm{d}x = -\frac{1}{b-1}x^{-(b-1)} \Big|_a^{\infty} = \frac{a^{1-b}}{b-1}.$$

Multi-Variable Calculus

1. If $f(x,y) = x^2$ and $R = \{0 < x < y < 1\}$, then
$$\iint\limits_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_x^1 x^2\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x^2 \left(\int_x^1 1\,\mathrm{d}y\right)\,\mathrm{d}x = \int_0^1 x^2(1-x)\,\mathrm{d}x = \frac{1}{12}.$$

2. If $f(x,y) = x^2$ and $R = \{0 < y < x < 1\}$, then
$$\iint\limits_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^x x^2\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x^2 \left(\int_0^x 1\,\mathrm{d}y\right)\,\mathrm{d}x = \int_0^1 x^2 \cdot x\,\mathrm{d}x = \frac{1}{4}.$$

3. If $f(x,y) = y^2$ and $R = \{0 < x < y < 1\}$, then
$$\iint\limits_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^y y^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 y^2 \left(\int_0^y 1\,\mathrm{d}x\right)\,\mathrm{d}y = \int_0^1 y^2 \cdot y\,\mathrm{d}x = \frac{1}{4}.$$

4. If $f(x,y) = xy$ and $R = \{0 < y < x < 1\}$, then
$$\iint\limits_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^x xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x \left(\int_0^x y\,\mathrm{d}y\right)\,\mathrm{d}x = \int_0^1 x \cdot \frac{1}{2}x^2\,\mathrm{d}x = \frac{1}{8}.$$

5. If $f(x,y) = x + y$ and $R = \{0 < x < y < 1\}$, then
$$\iint\limits_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^y (x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left(\frac{1}{2}x^2 + xy\right)\Bigg|_{x=0}^{x=y}\,\mathrm{d}x = \int_0^1 \frac{1}{2}y^2 + y^2\,\mathrm{d}x = \frac{1}{2}.$$

6. If $f(x,y) = e^{-2y}$ and $R = \{0 < x < 2y\}$, then
$$\iint\limits_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_0^{2y} e^{-2y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty 2ye^{-2y}\,\mathrm{d}y = \frac{1}{2}.$$

7. If $f(x,y) = \sqrt{x^2+y^2}$ and $R = \{x^2 + y^2 \le 1\}$, then making the change-of-variables to polar coordinates via $x = r\cos\theta$, $y = r\sin\theta$, $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ gives
$$\iint\limits_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \int_0^1 r \cdot r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi \int_0^1 r^2\,\mathrm{d}r = \frac{2}{3}\pi.$$
Note that R describes the region inside the circle of radius 1 centred at the origin. Thus, the limits for r and θ are $0 \le r \le 1$ and $0 \le \theta < 2\pi$. Furthermore, in polar coordinates, $\sqrt{x^2+y^2} = \sqrt{r^2} = r$.

8. If $f(x,y) = xy$ and $R = \{x^2 + y^2 \le 1,\ x > 0,\ y > 0\}$, then
$$\iint\limits_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^{\pi/2} r\cos\theta \cdot r\sin\theta \cdot r\,\mathrm{d}r\,\mathrm{d}\theta = \left(\int_0^1 r^3\,\mathrm{d}r\right) \cdot \left(\int_0^{\pi/2} \frac{1}{2}\sin(2\theta)\,\mathrm{d}\theta\right) = \frac{1}{8}.$$
Note that R describes the region inside the first quadrant of the circle of radius 1 centred at the origin. Thus, the limits for r and θ are $0 \le r \le 1$ and $0 < \theta < \pi/2$. Furthermore, in polar coordinates, $xy = r\cos\theta \cdot r\sin\theta = \frac{r^2}{2}\sin(2\theta)$.

Some Sums

1. Recall that if r satisfies $-1 < r < 1$, then

$$\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$$

gives the value of the geometric series. Thus,

$$\sum_{j=0}^{\infty} 3^{-j} = \sum_{j=0}^{\infty} (1/3)^j = \frac{1}{1-1/3} = \frac{3}{2}.$$

2. It is a fact that if r satisfies $-1 < r < 1$, then

$$\sum_{j=1}^{\infty} j r^j = \frac{r}{(1-r)^2}.$$

Here is how you prove this fact. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}r} r^j = j r^{j-1}.$$

Therefore,

$$\sum_{j=1}^{\infty} j r^j = r \sum_{j=1}^{\infty} j r^{j-1} = r \sum_{j=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}r} r^j.$$

If we now interchange the derivative and the summation, then we get

$$\sum_{j=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}r} r^j = \frac{\mathrm{d}}{\mathrm{d}r} \sum_{j=1}^{\infty} r^j.$$

However, if we notice that

$$\sum_{j=0}^{\infty} r^j = r^0 + \sum_{j=1}^{\infty} r^j = 1 + \sum_{j=1}^{\infty} r^j,$$

then we conclude that

$$\sum_{j=1}^{\infty} r^j = \sum_{j=0}^{\infty} r^j - 1 = \frac{1}{1-r} - 1 = \frac{r}{1-r}.$$

Putting this back in to the earlier expressions gives

$$\sum_{j=1}^{\infty} j r^j = r \cdot \frac{\mathrm{d}}{\mathrm{d}r} \sum_{j=1}^{\infty} r^j = r \cdot \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{r}{1-r} \right) = r \cdot \frac{1}{(1-r)^2} = \frac{r}{(1-r)^2}.$$

Hence, we find

$$\sum_{j=1}^{\infty} j 3^{-j} = \sum_{j=1}^{\infty} j (1/3)^j = \frac{1/3}{(1-1/3)^2} = \frac{3}{4}.$$

3. Recall that if $-\infty < x < \infty$, then the power series (i.e., Taylor series at 0 or Maclaurin series) for e^x is

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

Thus,

$$\sum_{j=0}^{\infty} \frac{3^{-j}}{j!} = \sum_{j=0}^{\infty} \frac{(1/3)^j}{j!} = e^{1/3}.$$