Math 302.102 Fall 2010
The Beta Function

The purpose of this exercise is to lead you through the verification that the density function of a Beta(a, b) random variable, namely

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 \le x \le 1,$$

is, in fact, a legitimate density. Suppose that

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x.$$

Our goal is to show that

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

(a) As a first step, we will need to prove that

$$B(a, b) = 2\int_0^{\pi/2} \cos^{2a-1}(\theta)\sin^{2b-1}(\theta)\,\mathrm{d}\theta.$$

This is done by making the substitution $x = \cos^2(\theta)$.

(b) Now consider

$$\Gamma(a)\Gamma(b) = \int_0^\infty u^{a-1}e^{-u}\,\mathrm{d}u \cdot \int_0^\infty v^{b-1}e^{-v}\,\mathrm{d}v = \int_0^\infty \int_0^\infty u^{a-1}v^{b-1}e^{-(u+v)}\,\mathrm{d}u\,\mathrm{d}v.$$

Change variables by letting $u = x^2$ and $v = y^2$ to show that

$$\Gamma(a)\Gamma(b) = 4\int_0^\infty \int_0^\infty x^{2a-1}y^{2b-1}e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$$

(c) Change to polar coordinates with $x = r\cos(\theta)$, $y = r\sin(\theta)$ for $0 \le r < \infty$, $0 \le \theta \le \pi/2$ to show that

$$\Gamma(a)\Gamma(b) = 4\int_0^\infty r^{2a+2b-2}e^{-r^2}r\,\mathrm{d}r \int_0^{\pi/2} \cos^{2a-1}(\theta)\sin^{2b-1}(\theta)\,\mathrm{d}\theta. \qquad (*)$$

(d) Let $t = r^2$ to show that

$$2\int_0^\infty r^{2a+2b-2}e^{-r^2}r\,\mathrm{d}r = \int_0^\infty t^{a+b-1}e^{-t}\,\mathrm{d}t = \Gamma(a+b).$$

(e) Combine **(a)** and **(d)** to conclude from $(*)$ that

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a, b)$$

as required.