Math 302.102 Fall 2010
Practice Problems for Midterm #2

Problem 1. Suppose that X is a continuous random variable with density function

$$f(x) = \begin{cases} \frac{3}{7}x^2, & 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Verify that f is, in fact, a legitimate density function.

(b) Compute $\mathbb{E}(X)$, the expected value (or mean or average) of X.

(c) Compute $\mathrm{Var}(X)$, the variance of X.

(d) Determine $F(x)$, the distribution function of X.

(e) Determine the *median* of X.

Problem 2. Suppose that X_1 and X_2 are independent continuous random variables each having common *distribution* function

$$F(x) = \begin{cases} 1 - xe^{-x} - e^{-x}, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

(a) Determine $f(x)$, their common density function.

(b) Suppose that $Y_1 = \min\{X_1, X_2\}$. Determine $f_{Y_1}(y)$, the density function of Y_1.

(c) Suppose that $Y_2 = \max\{X_1, X_2\}$. Determine $f_{Y_2}(y)$, the density function of Y_2.

(d) Let $Z_1 = Y_1^3$. Determine $f_{Z_1}(z)$, the density function of Z_1.

(e) Let $Z_2 = \sqrt{Y_2}$. Determine $f_{Z_2}(z)$, the density function of Z_2.

Problem 3. Suppose that X and Y are independent, continuous random variables. If the density function of X is $f_X(x) = xe^{-x}$ for $x \ge 0$, and the density function of Y is $f_Y(y) = e^{-y}$ for $y \ge 0$, use the law of total probability to determine $\mathbf{P}\{X < Y\}$. *Hint*: It is probably easier to condition on the value of X.

Problem 4. Suppose that X is a continuous random variable with distribution function $F(x)$ and density function $f(x)$. Suppose further that f is continuous. Use the law of the unconscious statistician to show that $\mathbb{E}[F(X)] = 1/2$.

Solutions

1. (a) Observe that $f(x) \geq 0$ for all $x \in \mathbb{R}$ and that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_1^2 \frac{3}{7}x^2\,\mathrm{d}x = \frac{1}{7}x^3\Big|_1^2 = \frac{8}{7} - \frac{1}{7} = 1$$

so that f is, in fact, a legitimate density.

1. (b) By definition,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x = \int_1^2 \frac{3}{7}x^3\,\mathrm{d}x = \frac{3}{28}x^4\Big|_1^2 = \frac{3}{28}(16-1) = \frac{45}{28}.$$

1. (c) We find

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x = \int_1^2 \frac{3}{7}x^4\,\mathrm{d}x = \frac{3}{35}x^5\Big|_1^2 = \frac{3}{35}(32-1) = \frac{93}{35}$$

so that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{93}{35} - \left[\frac{45}{28}\right]^2 = \frac{2037}{27440} \doteq 0.0742347.$$

1. (d) By definition, if $1 \leq x \leq 2$, then

$$F(x) = \int_{-\infty}^{x} f(u)\,\mathrm{d}u = \int_1^x \frac{3}{7}u^2\,\mathrm{d}u = \frac{1}{7}u^3\Big|_1^x = \frac{x^3}{7} - \frac{1}{7}.$$

1. (e) The median of X is that value a for which

$$\int_{-\infty}^{a} f(x)\,\mathrm{d}x = \frac{1}{2},$$

or equivalently, that value of a for which $F(a) = \mathbf{P}\{X \leq a\} = 1/2$. Thus, since we found F in **(d)**, we conclude that a satisfies

$$\frac{a^3}{7} - \frac{1}{7} = \frac{1}{2}$$

and so

$$a = \frac{9^{1/3}}{2^{1/3}}.$$

2. (a) If $x > 0$, then

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x) = xe^{-x}.$$

2. (b) If $y > 0$, then

$$\begin{aligned}\mathbf{P}\{Y_1 > y\} = \mathbf{P}\{\min\{X_1, X_2\} > y\} = \mathbf{P}\{X_1 > y, X_2 > y\} &= \mathbf{P}\{X_1 > y\}\,\mathbf{P}\{X_2 > y\} \\ &= [1 - \mathbf{P}\{X_1 \le y\}][1 - \mathbf{P}\{X_2 \le y\}] \\ &= [1 - F(y)]^2 \\ &= [ye^{-y} + e^{-y}]^2 \\ &= (y+1)^2 e^{-2y}\end{aligned}$$

so that

$$F_{Y_1}(y) = \mathbf{P}\{Y_1 \le y\} = 1 - \mathbf{P}\{Y_1 > y\} = 1 - (y+1)^2 e^{-2y}.$$

Thus,

$$f_{Y_1}(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_{Y_1}(y) = 2y(y+1)e^{-2y}.$$

2. (c) If $y > 0$, then

$$\begin{aligned}F_{Y_2}(y) = \mathbf{P}\{Y_2 \le y\} = \mathbf{P}\{\max\{X_1, X_2\} \le y\} &= \mathbf{P}\{X_1 \le y, X_2 \le y\} \\ &= \mathbf{P}\{X_1 \le y\}\,\mathbf{P}\{X_2 \le y\} \\ &= [F(y)]^2 \\ &= [1 - ye^{-y} - e^{-y}]^2.\end{aligned}$$

Thus,

$$f_{Y_2}(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_{Y_2}(y) = 2ye^{-y}(1 - ye^{-y} - e^{-y}).$$

2. (d) If $Z_1 = Y_1^3$, then for $z > 0$, the distribution function of Z_1 is

$$F_{Z_1}(z) = \mathbf{P}\{Z_1 \le z\} = \mathbf{P}\left\{Y_1^3 \le z\right\} = \mathbf{P}\left\{Y_1 \le z^{1/3}\right\} = \int_{-\infty}^{z^{1/3}} f_{Y_1}(y)\,\mathrm{d}y$$

so by the fundamental theorem of calculus, if $z > 0$, then

$$f_{Z_1}(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_{Z_1}(z) = f_{Y_1}(z^{1/3}) \frac{\mathrm{d}}{\mathrm{d}z} z^{1/3} = 2z^{1/3}(z^{1/3}+1)e^{-2z^{1/3}} \cdot \frac{1}{3} z^{-2/3} = \frac{2}{3}(1 + z^{-1/3})e^{-2z^{1/3}}.$$

2. (e) If $Z_2 = \sqrt{Y_2}$, then for $z > 0$, the distribution function of Z_2 is

$$F_{Z_2}(z) = \mathbf{P}\{Z_2 \le z\} = \mathbf{P}\left\{\sqrt{Y_2} \le z\right\} = \mathbf{P}\left\{Y_2 \le z^2\right\} = \int_{-\infty}^{z^2} f_{Y_2}(y)\,\mathrm{d}y$$

so by the fundamental theorem of calculus, if $z > 0$, then

$$f_{Z_2}(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_{Z_2}(z) = f_{Y_2}(z^2) \frac{\mathrm{d}}{\mathrm{d}z} z^2 = 2z^2 e^{-z^2}(1 - z^2 e^{-z^2} - e^{-z^2}) \cdot 2z = 4z^3 e^{-z^2}(1 - z^2 e^{-z^2} - e^{-z^2}).$$

3. By the law of total probability,

$$
\begin{aligned}
\mathbf{P}\{X < Y\} = \int_{-\infty}^{\infty} \mathbf{P}\{Y > x\}\, f_X(x)\,\mathrm{d}x &= \int_0^{\infty} \left[\int_x^{\infty} e^{-y}\,\mathrm{d}y\right] xe^{-x}\,\mathrm{d}x \\
&= \int_0^{\infty} \left[e^{-x}\right] xe^{-x}\,\mathrm{d}x \\
&= \int_0^{\infty} xe^{-2x}\,\mathrm{d}x \\
&= \frac{1}{4}\int_0^{\infty} ue^{-u}\,\mathrm{d}u \\
&= \frac{1}{4}.
\end{aligned}
$$

(Note that this final integral equals 1 since it represents the total area under a density curve—the density for X, in fact.)

4. By the law of the unconscious statistician, we have

$$\mathbb{E}[F(X)] = \int_{-\infty}^{\infty} F(x)f(x)\,\mathrm{d}x.$$

If we make the change of variables $u = F(x)$, then $\mathrm{d}u = F'(x)\,\mathrm{d}x$. But we know that $F' = f$ so that $\mathrm{d}u = f(x)\,\mathrm{d}x$. Now for the limits of integration. Since $F(x) \to 1$ as $x \to \infty$ and since $F(x) \to 0$ as $x \to -\infty$, we find

$$\int_{-\infty}^{\infty} F(x)f(x)\,\mathrm{d}x = \int_0^1 u\,\mathrm{d}u = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}$$

as required.