Math 302.102 Fall 2010
Extra Exercises

Exercise. Due to a fundamental problem of measurement, the location and position of a very small particle cannot be predicted with certainty. (This is the Heisenberg uncertainty principle.) The only thing that can be known is the particle's *wave function* ψ and hence a probability density function $f(x) = |\psi(x)|^2$. This wave function ψ is a time-weighted superposition of eigenfunctions (*stationary states*) of the *Hamiltonian operator* $\mathcal{H}$, the instrument that observes total energy. For example, the state of lowest energy (the *ground state*) of a quantum particle trapped in the potential free interval $[0,1]$ is $\psi(x) = \sqrt{2}\sin(\pi x)$. Determine the average position of a particle in this state.

Solution. Since the probability density function is the square of the wave function, we have $f(x) = |\psi(x)|^2 = \left[\sqrt{2}\sin(\pi x)\right]^2 = 2\sin^2(\pi x)$ for $0 \le x \le 1$. Thus, the expected (or mean or average) position of a particle in this state is

$$\int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x = 2\int_0^1 x\sin^2(\pi x)\,\mathrm{d}x = \cdots = \frac{1}{2}.$$

(Use integration-by-parts to fill in the missing $\cdots$ step.)

Exercise. Recall that the Gamma function is defined for $p > 0$ by

$$\Gamma(p) = \int_0^{\infty} x^{p-1}e^{-x}\,\mathrm{d}x.$$

Use integration-by-parts with to show that $\Gamma(p+1) = p\Gamma(p)$. In other words, consider

$$\Gamma(p+1) = \int_0^{\infty} x^p e^{-x}\,\mathrm{d}x \tag{$*$}$$

and use integration-by-parts once with $u = x^p$ and $\mathrm{d}v = e^{-x}\,\mathrm{d}v$ to show that $\Gamma(p+1) = p\Gamma(p)$. As a result, we can view the Gamma function as a natural generalization of the factorial function. If $k \ge 1$ is an integer, then by repeated use of $(*)$ implies

$$\begin{aligned}\Gamma(k+1) = k\Gamma(k) = k(k-1)\Gamma(k-2) &= k(k-1)(k-2)\Gamma(k-3)\\ &\vdots\\ &= k(k-1)(k-2)\cdots 2\Gamma(1)\\ &= k(k-1)(k-2)\cdots 2\cdot 1\\ &= k!\end{aligned}$$

using the fact at the last step that

$$\Gamma(1) = \int_0^{\infty} e^{-x}\,\mathrm{d}x = 1.$$

Exercise. Suppose that $X \sim \text{Beta}(a,b)$. Show directly and quickly (without using moment generating functions) that

$$\mathbb{E}(X^k) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a+k+b)} = \frac{\Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+k+b)}.$$

Note that you can now use the result $(*)$ of the earlier exercise to obtain a recursive formula for $\mathbb{E}(X^k)$, namely

$$\mathbb{E}(X^k) = \frac{a+k-1}{a+b+k-1}\mathbb{E}(X^{k-1}).$$

Exercise. Data on the life span T (in hours) of 60W hotel hallway lightbulbs provided by General Electric is given below.

hours	400	500	600	700	800	900	1000	1100	1200	1300	1400
fail	0%	2%	5%	10%	20%	30%	50%	70%	80%	90%	95%

This is real data. Graph it. You will notice a striking similarity to the distribution function of an exponential random variable with parameter λ. Based on the data, estimate λ. Superimpose the graph of this distribution function with the estimated value of λ over the actual data.

Exercise. Amateur astronomers have been obtaining spectacular photographs with small telescopes fitted with inexpensive, low-pixel-count CCD cameras by taking multiple ($n > 50$) exposures, then superimposing these exposures using photo manipulation software. Their results are comparable to older photographs taken with huge telescopes. Why does this technique produce such stunning photos? See the work of Thierry Legault online at

`http://legault.perso.sfr.fr/`

for many stunning examples.

Exercise. Many of the distributions that we study are named for the simple reason that they are encountered frequently. In fact, many of them appear *naturally.* Suppose that shots are being fired at the bullseye of a standard target marked in the usual way with concentric circles. The errors in the horizontal and vertical distances are independent and each normally distributed with mean 0 and variance σ^2. Show that the error in terms of the distance r from the centre of the bullseye has a *Rayleigh* distribution with

$$F(r) = 1 - e^{-r^2/2\sigma^2}$$

for $r > 0$. Formally, suppose that $X \sim \mathcal{N}(0,\sigma^2)$ and $Y \sim \mathcal{N}(0,\sigma^2)$ are independent. Let $R = \sqrt{X^2+Y^2}$ and show that the distribution function of R is $F(r)$.