Math 302.102 Fall 2010
Solutions to Assignment #7

1. (a) We must choose the value of c so that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$. Since

$$\int_a^b\int_a^y (y-x)\,\mathrm{d}x\,\mathrm{d}y = -\int_a^b \frac{(y-x)^2}{2}\Big|_{x=a}^{x=y}\mathrm{d}y = \int_a^b \frac{(y-a)^2}{2}\,\mathrm{d}y = \frac{(y-a)^3}{6}\Big|_{y=a}^{y=b} = \frac{(b-a)^3}{6}$$

we conclude that

$$c = \frac{6}{(b-a)^3}.$$

1. (b) By definition,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \int_a^y c(y-x)\,\mathrm{d}x = -\frac{c(y-x)^2}{2}\Big|_{x=a}^{x=y} = \frac{c(y-a)^2}{2} = \frac{3(y-a)^2}{(b-a)^3}$$

for $a \le y \le b$. Thus,

$$\begin{aligned}\mathbb{E}(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,\mathrm{d}y = \frac{3}{(b-a)^3}\int_a^b y(y-a)^2\,\mathrm{d}y &= \frac{3}{(b-a)^3}\int_a^b (y^3 - 2ay^2 + a^2y)\,\mathrm{d}y\\ &= \frac{3}{(b-a)^3}\left[\frac{y^4}{4} - \frac{2ay^3}{3} + \frac{a^2y^2}{2}\right]_a^b\\ &= \frac{3}{(b-a)^3}\left[\frac{b^4}{4} - \frac{2ab^3}{3} + \frac{a^2b^2}{2} - \frac{a^4}{4} + \frac{2a^4}{3} - \frac{a^4}{2}\right]\\ &= \frac{3b^4 - 8ab^3 + 6a^2b^2 - a^4}{4(b-a)^3}\\ &= \frac{(3b+a)(b-a)^3}{4(b-a)^3}\\ &= \frac{3b+a}{4}.\end{aligned}$$

1. (c) By definition,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_x^b c(y-x)\,\mathrm{d}y = \frac{c(y-x)^2}{2}\Big|_{y=x}^{y=b} = \frac{c(b-x)^2}{2} = \frac{3(b-x)^2}{(b-a)^3}$$

for $a \le y \le b$. Thus,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \frac{3}{(b-a)^3}\int_a^b x(b-x)^2\,\mathrm{d}x = \frac{3}{(b-a)^3}\int_a^b (x^3 - 2bx^2 + b^2x)\,\mathrm{d}x.$$

Notice that this is the same computation as $\mathbb{E}(Y)$ except that b appears in the integrand instead of a. Hence, if we write

$$\frac{3}{(b-a)^3}\int_a^b (x^3 - 2bx^2 + b^2x)\,\mathrm{d}x = \frac{3}{(a-b)^3}\int_b^a (x^3 - 2bx^2 + b^2x)\,\mathrm{d}x,$$

(one minus sign from switching the limits of integration and one minus sign from changing $(b-a)^3$ to $(a-b)^3$), then it is the same computation as above with a and b switched. Thus,

$$\mathbb{E}(X) = \frac{3a+b}{4}.$$

2. (a) By definition,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_{x}^{\infty} 2e^{-x-y}\,\mathrm{d}y = 2e^{-x}\int_{x}^{\infty} e^{-y}\,\mathrm{d}y = 2e^{-x}e^{-x} = 2e^{-2x}$$

for $x > 0$. By definition,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \int_{0}^{y} 2e^{-x-y}\,\mathrm{d}x = 2e^{-y}\int_{0}^{y} e^{-x}\,\mathrm{d}x = 2e^{-y}(1-e^{-y})$$

for $y > 0$.

2. (b) Since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, we conclude that X and Y are not independent.

2. (c) By definition,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_{0}^{\infty} x \cdot 2e^{-2x}\,\mathrm{d}x = \int_{0}^{\infty} 2xe^{-2x}\,\mathrm{d}x = \frac{1}{2}.$$

By definition,

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,\mathrm{d}y = \int_{0}^{\infty} y \cdot 2e^{-y}(1-e^{-y})\,\mathrm{d}y = \int_{0}^{\infty} 2ye^{-y}\,\mathrm{d}y - \int_{0}^{\infty} 2ye^{-2y}\,\mathrm{d}y = 2 - \frac{1}{2} = \frac{3}{2}.$$