Math 302.102 Fall 2010
Solutions to Assignment #6

1. The definition of conditional probability implies that

$$\mathbf{P}\left\{X > t+s \,|\, X > t\right\} = \frac{\mathbf{P}\left\{X > t+s, X > t\right\}}{\mathbf{P}\left\{X > t\right\}} = \frac{\mathbf{P}\left\{X > t+s\right\}}{\mathbf{P}\left\{X > t\right\}}$$

since the only way for both $\{X > t+s\}$ and $\{X > t\}$ to happen is if $\{X > t+s\}$ happens. Since $X \sim \text{Exp}(\lambda)$, we find that if $a > 0$, then

$$\mathbf{P}\left\{X > a\right\} = \int_a^\infty \lambda e^{\lambda x}\,\mathrm{d}x = -e^{-\lambda x}\Big|_a^\infty = e^{-\lambda a}.$$

Therefore,

$$\mathbf{P}\left\{X > t+s \,|\, X > t\right\} = \frac{\mathbf{P}\left\{X > t+s\right\}}{\mathbf{P}\left\{X > t\right\}} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbf{P}\left\{X > s\right\}.$$

Equivalently, Bayes' rule implies that

$$\mathbf{P}\left\{X > t+s \,|\, X > t\right\} = \frac{\mathbf{P}\left\{X > t \,|\, X > t+s\right\}\mathbf{P}\left\{X > t+s\right\}}{\mathbf{P}\left\{X > t\right\}} = \frac{\mathbf{P}\left\{X > t+s\right\}}{\mathbf{P}\left\{X > t\right\}}$$

since $\mathbf{P}\left\{X > t \,|\, X > t+s\right\} = 1$; that is, if know X is at least $t+s$, then we know with certainty that X is at least t. We find, as above,

$$\mathbf{P}\left\{X > t+s \,|\, X > t\right\} = \frac{\mathbf{P}\left\{X > t+s\right\}}{\mathbf{P}\left\{X > t\right\}} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbf{P}\left\{X > s\right\}.$$

2. If $X \sim \text{Unif}(0,1)$, then

$$f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) We find

$$\mu = \mathbb{E}(X) = \int_{-\infty}^\infty x f(x)\,\mathrm{d}x = \int_0^1 x \cdot 1\,\mathrm{d}x = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$$

and therefore

$$\sigma^2 = \text{var}(X) = \int_{-\infty}^\infty (x-\mu)^2 f(x)\,\mathrm{d}x = \int_0^1 (x-1/2)^2 \cdot 1\,\mathrm{d}x = \frac{1}{3}(x-1/2)^3\Big|_0^1 = \frac{(1/2)^3 - (-1/2)^3}{3} = \frac{1}{12}.$$

Equivalently, $\text{var}(X)$ can be determined by first computing

$$\mathbb{E}(X^2) = \int_{-\infty}^\infty x^2 f(x)\,\mathrm{d}x = \int_0^1 x^2 \cdot 1\,\mathrm{d}x = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

and noting that

$$\sigma^2 = \text{var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

(b) We begin by noting that

$$\mathbf{P}\left\{\mu - 2\sigma < X < \mu + 2\sigma\right\} = \mathbf{P}\left\{\frac{1}{2} - \frac{1}{\sqrt{3}} < X < \frac{1}{2} + \frac{1}{\sqrt{3}}\right\} = \int_{\frac{1}{2}-\frac{1}{\sqrt{3}}}^{\frac{1}{2}+\frac{1}{\sqrt{3}}} f(x)\,\mathrm{d}x.$$

Observe that

$$\frac{1}{2} - \frac{1}{\sqrt{3}} < 0 \quad\text{and}\quad \frac{1}{2} + \frac{1}{\sqrt{3}} > 1.$$

Thus, since $f(x) = 1$ only when $0 \le x \le 1$, we see that

$$\begin{aligned}
\int_{\frac{1}{2}-\frac{1}{\sqrt{3}}}^{\frac{1}{2}+\frac{1}{\sqrt{3}}} f(x)\,\mathrm{d}x &= \int_{\frac{1}{2}-\frac{1}{\sqrt{3}}}^{0} f(x)\,\mathrm{d}x + \int_0^1 f(x)\,\mathrm{d}x + \int_1^{\frac{1}{2}+\frac{1}{\sqrt{3}}} f(x)\,\mathrm{d}x \\
&= \int_{\frac{1}{2}-\frac{1}{\sqrt{3}}}^{0} 0\,\mathrm{d}x + \int_0^1 1\,\mathrm{d}x + \int_1^{\frac{1}{2}+\frac{1}{\sqrt{3}}} 0\,\mathrm{d}x \\
&= 0 + \int_0^1 1\,\mathrm{d}x + 0 \\
&= 1.
\end{aligned}$$

Chebychev's inequality states that $\mathbf{P}\left\{\mu - 2\sigma < X < \mu + 2\sigma\right\} \ge 0.75$; in other words, the area under any density curve within two standard deviations of the mean is at least 0.75. In this example, the area is 1 which, as promised by Chebychev's inequality, is at least 0.75.

3. Suppose that X is a random variable with density function

$$f_X(x) = \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}}, \quad 0 < x < \infty.$$

Let $Y = 1/(1+\frac{m}{n}X)$ so that if $0 \le y \le 1$, then the distribution function of Y is

$$\begin{aligned}
F_Y(y) &= \mathbf{P}\left\{Y \le y\right\} = \mathbf{P}\left\{1/(1+\tfrac{m}{n}X) \le y\right\} = \mathbf{P}\left\{1+\tfrac{m}{n}X \ge 1/y\right\} = \mathbf{P}\left\{X \ge \tfrac{n}{m}(1/y-1)\right\} \\
&= 1 - \mathbf{P}\left\{X \le \tfrac{n}{m}(1/y-1)\right\} \\
&= 1 - \int_0^{\frac{n}{m}(1/y-1)} \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}}\,dx.
\end{aligned}$$

Taking derivatives with respect to y gives

$$\begin{aligned}
f_Y(y) &= \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{(\frac{n}{m}(1/y-1))^{m/2-1}}{(1+\frac{m\frac{n}{m}(1/y-1)}{n})^{(m+n)/2}} \cdot \frac{n}{my^2} \\
&= \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}\left(\frac{n}{m}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{(1/y-1)^{m/2-1}}{(1/y)^{(m+n)/2}} \cdot \frac{1}{y^2} \\
&= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} y^{m/2+n/2} y^{-2} y^{1-m/2} (1-y)^{m/2-1} \\
&= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} y^{n/2-1} (1-y)^{m/2-1}
\end{aligned}$$

for $0 \le y \le 1$. We recognize that this is the density of a Beta$(n/2, m/2)$ random variable, and so we conclude that $Y = 1/(1+\frac{m}{n}X) \sim \text{Beta}(n/2, m/2)$.

4. Suppose that X is a random variable with density function

$$f_X(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, \quad -\infty < x < \infty.$$

Let $Y = X^2$. If $y \geq 0$, then the distribution function of Y is given by

$$\begin{aligned} F_Y(y) = \mathbf{P}\{Y \leq y\} = \mathbf{P}\{X^2 \leq y\} = \mathbf{P}\{-\sqrt{y} \leq X \leq \sqrt{y}\} &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx \\ &= \int_0^{\sqrt{y}} f_X(x)\,dx - \int_0^{-\sqrt{y}} f_X(x)\,dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$\begin{aligned} f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} &= \frac{1}{2\sqrt{y}}\,(\,f_X(\sqrt{y}) + f_X(-\sqrt{y})\,) \\ &= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n y}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{y}{n}\right)^{-(n+1)/2} \end{aligned}$$

for $y \geq 0$. Equivalently, if use the fact that $\Gamma(1/2) = \sqrt{\pi}$, then for $y \geq 0$ we can express f_Y as

$$f_Y(y) = \frac{\Gamma(\frac{1+n}{2})\left(\frac{1}{n}\right)^{1/2}}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})} \cdot \frac{y^{1/2-1}}{\left(1 + \frac{y}{n}\right)^{(1+n)/2}}.$$

(Notice that this density is the same as the one given in the previous problem with $m = 1$.)

5. If we let $Y = M - 3$, then we are told that $Y \sim \text{Exp}(1/2)$ so that $\mathbb{E}(Y) = 2$ and $\text{var}(Y) = 4$. This was done in class. If $Y \sim \text{Exp}(\lambda)$, then $\mathbb{E}(Y) = 1/\lambda$ and $\text{var}(Y) = 1/\lambda^2$. Hence, if $\mathbb{E}(Y) = 2$, then it must be the case that $\lambda = 1/2$.

(a) We find $\mathbb{E}(Y) = \mathbb{E}(M - 3) = \mathbb{E}(M) - 3$ so that $\mathbb{E}(M) = \mathbb{E}(Y) + 3 = 2 + 3 = 5$. We also find that $\text{var}(Y) = \text{var}(M - 3) = \text{var}(M) = 4$.

(b) Since $M = \log X$, we can write $Y = \log(X) - 3$ where $Y \sim \text{Exp}(1/2)$. For $\log(x) - 3 \geq 0$, or equivalently, for $x \geq e^3$, the distribution function of X is

$$\begin{aligned} F_X(x) = \mathbf{P}\{X \leq x\} = \mathbf{P}\{\log(X) - 3 \leq \log(x) - 3\} &= \mathbf{P}\{Y \leq \log(x) - 3\} \\ &= \int_{-\infty}^{\log(x)-3} f_Y(y)\,\mathrm{d}y \\ &= \int_0^{\log(x)-3} \frac{1}{2} e^{-y/2}\,\mathrm{d}y \\ &= 1 - e^{-1/2(\log(x)-3)} \\ &= 1 - e^{3/2} x^{-1/2}. \end{aligned}$$

Thus, the density function of X is

$$f_X(x) = \begin{cases} \dfrac{e^{3/2}}{2} x^{-3/2}, & x \geq e^3, \\ 0, & x < e^3. \end{cases}$$

(c) Let M_1 and M_2 denote the magnitudes of the two earthquakes so that $Y_1 = M_1 - 3$ and $Y_2 = M_2 - 3$ are independent Exp(1/2) random variables. We are interested in computing $\mathbf{P}\{\min\{M_1, M_2\} > 4\}$. However, we don't know the distributions of M_1 and M_2. Instead, we observe that

$$\min\{M_1, M_2\} > 4 \text{ if and only if } \{M_1 - 3, M_2 - 3\} > 4 - 3,$$

That is, $\mathbf{P}\{\min\{M_1, M_2\} > 4\} = \mathbf{P}\{\min\{Y_1, Y_2\} > 1\}$. Hence,

$$\begin{aligned} \mathbf{P}\{\min\{Y_1, Y_2\} > 1\} = \mathbf{P}\{Y_1 > 1, Y_2 > 1\} = \mathbf{P}\{Y_1 > 1\}\mathbf{P}\{Y_2 > 1\} &= [\mathbf{P}\{Y_1 > 1\}]^2 \\ &= \left[\int_1^\infty \frac{1}{2} e^{-y/2}\, \mathrm{d}y\right]^2 \\ &= [e^{-1/2}]^2 \\ &= e^{-1}. \end{aligned}$$

6. Let X_i be the lifetime of the ith component so that $X_i \sim \text{Exp}(\lambda_i)$ where λ_i is given in the diagram in the problem. If $Y_1 = \min\{X_1, X_2, X_3\}$, $Y_2 = \min\{X_4, X_5\}$, and $Y = \max\{Y_1, Y_2\}$, then the expected lifetime of the circuit is given by $\mathbb{E}(Y)$. We begin by finding the distribution function of Y_1. That is, if $y \geq 0$, then

$$\begin{aligned} F_{Y_1}(y) = \mathbf{P}\{Y_1 \leq y\} = 1 - \mathbf{P}\{Y_1 > y\} &= 1 - \mathbf{P}\{\min\{X_1, X_2, X_3\} > y\} \\ &= 1 - \mathbf{P}\{X_1 > y, X_2 > y, X_3 > y\} \\ &= 1 - \mathbf{P}\{X_1 > y\}\mathbf{P}\{X_2 > y\}\mathbf{P}\{X_3 > y\}. \end{aligned}$$

If $X_i \sim \text{Exp}(\lambda_i)$, then

$$\mathbf{P}\{X_i > y\} = \int_y^\infty \lambda e^{\lambda x}\, \mathrm{d}x = -e^{-\lambda x}\Big|_y^\infty = e^{-\lambda y}.$$

(This same calculation was done in Problem 1.) Thus, if $y \geq 0$, then

$$F_{Y_1}(y) = 1 - e^{-\lambda_1 y} e^{-\lambda_2 y} e^{-\lambda_3 y} = 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)y}.$$

The distribution function of Y_2 is found in exactly the same manner. That is, if $y \geq 0$, then

$$\begin{aligned} F_{Y_2}(y) = \mathbf{P}\{Y_2 \leq y\} = 1 - \mathbf{P}\{Y_2 > y\} &= 1 - \mathbf{P}\{\min\{X_4, X_5\} > y\} \\ &= 1 - \mathbf{P}\{X_4 > y, X_5 > y\} \\ &= 1 - \mathbf{P}\{X_4 > y\}\mathbf{P}\{X_5 > y\} \\ &= 1 - e^{-\lambda_4 y} e^{-\lambda_5 y} \\ &= 1 - e^{-(\lambda_4+\lambda_5)y}. \end{aligned}$$

For $y \geq 0$, the distribution function of Y is

$$\begin{aligned} F_Y(y) = \mathbf{P}\{Y \leq y\} = \mathbf{P}\{\max\{Y_1, Y_2\} \leq y\} &= \mathbf{P}\{Y_1 \leq y, Y_2 \leq y\} \\ &= \mathbf{P}\{Y_1 \leq y\}\mathbf{P}\{Y_2 \leq y\} \\ &= \left[1 - e^{-(\lambda_1+\lambda_2+\lambda_3)y}\right]\left[1 - e^{-(\lambda_4+\lambda_5)y}\right] \\ &= 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)y} - e^{-(\lambda_4+\lambda_5)y} + e^{-(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)y} \end{aligned}$$

so that the density function of Y is

$$f_Y(y) = (\lambda_1+\lambda_2+\lambda_3)e^{-(\lambda_1+\lambda_2+\lambda_3)y}+(\lambda_4+\lambda_5)e^{-(\lambda_4+\lambda_5)y}-(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)e^{-(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)y}$$

for $y \geq 0$. Since we now know the density function for Y, we can compute $\mathbb{E}(Y)$, that is

$$\begin{aligned}
\mathbb{E}(Y) &= \int_{-\infty}^{\infty} y f_Y(y)\,\mathrm{d}y \\
&= (\lambda_1+\lambda_2+\lambda_3)\int_0^{\infty} y e^{-(\lambda_1+\lambda_2+\lambda_3)y}\,\mathrm{d}y + (\lambda_4+\lambda_5)\int_0^{\infty} y e^{-(\lambda_4+\lambda_5)y}\,\mathrm{d}y \\
&\quad - (\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)\int_0^{\infty} y e^{-(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)y}\,\mathrm{d}y \\
&= \frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{1}{\lambda_4+\lambda_5} - \frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5}.
\end{aligned}$$

Note that

$$\int_0^{\infty} \lambda^2 x e^{-\lambda x}\,\mathrm{d}x = 1$$

since it is the integral of the density of a Gamma$(2, \lambda)$ random variable. Thus,

$$\int_0^{\infty} \lambda x e^{-\lambda x}\,\mathrm{d}x = \frac{1}{\lambda}.$$

Finally, we can substitute in the values of λ_i, namely $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, $\lambda_3 = 0.3$, $\lambda_4 = 0.1$, $\lambda_5 = 0.1$, to conclude that

$$\mathbb{E}(Y) = \frac{1}{0.3+0.4+0.3} + \frac{1}{0.1+0.1} - \frac{1}{0.3+0.4+0.3+0.1+0.1} = \frac{31}{6}.$$

7. If $X \sim$ Gamma(α, λ), then

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

for $x \geq 0$ and so

$$m(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,\mathrm{d}x = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} e^{tx} x^{\alpha-1} e^{-\lambda x}\,\mathrm{d}x = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha-1} e^{-(\lambda-t)x}\,\mathrm{d}x.$$

We recognize the integral as the integral of a gamma density function with parameters α and $\lambda - t$. Thus,

$$\int_0^{\infty} x^{\alpha-1} e^{-(\lambda-t)x} = \frac{\Gamma(\alpha)}{(\lambda-t)^{\alpha}}$$

and so

$$m(t) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(\lambda-t)^{\alpha}} = \frac{\lambda^{\alpha}}{(\lambda-t)^{\alpha}}.$$