Math 302.102 Fall 2010
Solutions to Assignment #4

1. **(a)** We find

$$\mathbf{P}\{0 < X < 2\} = \int_0^2 f(x)\,\mathrm{d}x = \int_1^2 x^{-2}\,\mathrm{d}x = -x^{-1}\Big|_1^2 = 1 - \frac{1}{2} = \frac{1}{2}.$$

(b) We find

$$\mathbf{P}\{0 < X < 2\} = \int_0^2 f(x)\,\mathrm{d}x = \int_0^2 7e^{-7x}\,\mathrm{d}x = -e^{-7x}\Big|_0^2 = 1 - e^{-14}.$$

(c) We find

$$\mathbf{P}\{0 < X < 2\} = \int_0^2 f(x)\,\mathrm{d}x = \int_0^1 \frac{e}{e^2-5}x^2e^{-x}\,\mathrm{d}x = -\frac{e}{e^2-5}\cdot e^{-x}\left[x^2+2x+2\right]\Big|_0^1 = \frac{2e-5}{e^2-5}.$$

(d) We find

$$\mathbf{P}\{0 < X < 2\} = \int_0^2 f(x)\,\mathrm{d}x = \int_0^2 xe^{-x}\,\mathrm{d}x = \left[-xe^{-x} - e^{-x}\right]\Big|_0^2 = 1 - 3e^{-2}.$$

2. **(a)** Notice that we must necessarily have $0 < a < 4$. Since

$$\mathbf{P}\{X \le a\} = \int_0^a \frac{1}{8}x\,\mathrm{d}x = \frac{x^2}{16}\Big|_0^a = \frac{a^2}{16},$$

we find that in order for $\mathbf{P}\{X \le a\} = 1/2$ we must have

$$\frac{a^2}{16} = \frac{1}{2}$$

implying that $a^2 = 8$. The restriction that $a > 0$ implies that the unique value of a such that $\mathbf{P}\{X \le a\} = 1/2$ is $a = \sqrt{8}$.

(b) As in **(a)**, we must necessarily have $0 < a < 4$. Since

$$\mathbf{P}\{X \ge a\} = \int_a^4 \frac{1}{8}x\,\mathrm{d}x = \frac{x^2}{16}\Big|_a^4 = 1 - \frac{a^2}{16},$$

we find that in order for $\mathbf{P}\{X \le a\} = 1/2$ we must have

$$1 - \frac{a^2}{16} = \frac{1}{4}$$

implying that $a^2 = 12$. The restriction that $a > 0$ implies that the unique value of a such that $\mathbf{P}\{X \ge a\} = 1/4$ is $a = \sqrt{12}$.

3. Recall that the distribution function F of a random variable is defined as

$$F(x) = \mathbf{P}\{X \le x\}.$$

Since X is a continuous random variable, we know that $\mathbf{P}\{X = x\} = 0$ for any $x \in \mathbb{R}$ so that

$$\mathbf{P}\{X < x\} = \mathbf{P}\{X \le x\} = F(x).$$

(a) We find

$$\mathbf{P}\{X \le 1\} = F(1) = \frac{1}{8}(1)^3 = \frac{1}{8}.$$

(b) We find

$$\mathbf{P}\{0.5 \le X \le 1.5\} = \mathbf{P}\{X \le 1.5\} - \mathbf{P}\{X < 0.5\} = F(1.5) - F(0.5) = \frac{1}{8}\left[(1.5)^3 - (0.5)^3\right] = \frac{13}{32}.$$

(c) Notice that we must necessarily have $0 < a < 2$. Since

$$\mathbf{P}\{X \le a\} = F(a) = \frac{a^3}{8},$$

we find that in order for $\mathbf{P}\{X \le a\} = 1/2$ we must have

$$\frac{a^3}{8} = \frac{1}{2}$$

implying that $a^3 = 4$. Thus, the unique value of a such that $\mathbf{P}\{X \le a\} = 1/2$ is $a = 4^{1/3} = \sqrt[3]{4}$.

(d) As in **(c)**, we must necessarily have $0 < a < 2$. Since

$$\mathbf{P}\{X \ge a\} = 1 - \mathbf{P}\{X < a\} = 1 - F(a) = 1 - \frac{a^3}{8},$$

we find that in order for $\mathbf{P}\{X \ge a\} = 1/4$ we must have

$$1 - \frac{a^3}{8} = \frac{1}{4}$$

implying that $a^3 = 6$. Thus, the unique value of a such that $\mathbf{P}\{X \ge a\} = 1/4$ is $a = 6^{1/3} = \sqrt[3]{6}$.

4. Let

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, \mathrm{d}t$$

denote the distribution function of a normally distributed random variable. Hence, we find the following.

(a) $\mathbf{P}\{X > 1\} = 1 - \mathbf{P}\{X \le 1\} = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$.

(b) $\mathbf{P}\{X < 1\} = \mathbf{P}\{X \le 1\} = \Phi(1) = 0.8413$.

(c) $\mathbf{P}\{X \le 1\} = \Phi(1) = 0.8413$.

(d) $\mathbf{P}\{-1 \le X \le 1\} = \mathbf{P}\{X \le 1\} - \mathbf{P}\{X < -1\} = \mathbf{P}\{X \le 1\} - \mathbf{P}\{X \le -1\} = \Phi(1) - \Phi(-1) = (0.8413) - (1 - 0.8413) = 0.8413 - 0.1587 = 0.6826$.

(e) $\mathbf{P}\{X \le 2\} = \Phi(2) = 0.9772.$

(f) $\mathbf{P}\{X \ge -2\} = \mathbf{P}\{X \le 2\} = \Phi(2) = 0.9772.$

(g) $\mathbf{P}\{-2 \le X < 3\} = \mathbf{P}\{X < 3\} - \mathbf{P}\{X < -2\} = \mathbf{P}\{X \le 3\} - \mathbf{P}\{X \le -2\} = \Phi(3) - \Phi(-2) = (0.9987) - (1 - 0.9772) = 0.9987 - 0.0228 = 0.9759.$

(h) $\mathbf{P}\{-1 \le X \le 3\} = \mathbf{P}\{X < 3\} - \mathbf{P}\{X < -1\} = \mathbf{P}\{X \le 3\} - \mathbf{P}\{X \le -1\} = \Phi(3) - \Phi(-1) = 0.9987 - 0.1587 = 0.8400.$

5. **(a)** Let A_j, $j = 1, 2, 3$, denote the event that the lifetime of the jth TV lasts for at least two years. Therefore, since the TVs are selected at random and the TV lifetimes are independent, we find

$$\mathbf{P}\{A_1\} = \mathbf{P}\{A_2\} = \mathbf{P}\{A_3\} = \mathbf{P}\{X \ge 2\} = \int_2^\infty f(x)\,\mathrm{d}x = \int_2^\infty \frac{1}{2}e^{-x/2}\,\mathrm{d}x = -e^{-x/2}\Big|_2^\infty = \frac{1}{e}.$$

Hence, the probability that all three TVs last for at least two years is

$$\mathbf{P}\{A_1 \cap A_2 \cap A_3\} = \mathbf{P}\{A_1\}\,\mathbf{P}\{A_2\}\,\mathbf{P}\{A_3\} = \left(\frac{1}{e}\right)^3 = e^{-3}.$$

(b) Let B_j, $j = 1, 2, 3$, denote the event that the lifetime of the jth TV lasts for less than one year. Therefore, since the TVs are selected at random and the TV lifetimes are independent, we find

$$\mathbf{P}\{B_1\} = \mathbf{P}\{B_2\} = \mathbf{P}\{B_3\} = \mathbf{P}\{X < 1\} = \int_{-\infty}^1 f(x)\,\mathrm{d}x = \int_0^1 \frac{1}{2}e^{-x/2}\,\mathrm{d}x = -e^{-x/2}\Big|_0^1 = 1 - e^{-1/2}.$$

Hence, the probability that exactly one TV last for less than one year

$$\begin{aligned}
&\mathbf{P}\{\text{exactly one TV lasts for less than one year}\}\\
&\quad= \mathbf{P}\{B_1 \cap B_2^c \cap B_3^c \text{ or} B_1^c \cap B_2 \cap B_3^c \text{ or } B_1^c \cap B_2^c \cap B_3\}\\
&\quad= \mathbf{P}\{B_1 \cap B_2^c \cap B_3^c\} + \mathbf{P}\{B_1^c \cap B_2 \cap B_3^c\} + \mathbf{P}\{B_1^c \cap B_2^c \cap B_3\}\\
&\quad= \mathbf{P}\{B_1\}\,\mathbf{P}\{B_2^c\}\,\mathbf{P}\{B_3^c\} + \mathbf{P}\{B_1^c\}\,\mathbf{P}\{B_2\}\,\mathbf{P}\{B_3^c\} + \mathbf{P}\{B_1^c\}\,\mathbf{P}\{B_2^c\}\,\mathbf{P}\{B_3\}\\
&\quad= (1 - e^{-1/2})(e^{-1/2})^2 + (1 - e^{-1/2})(e^{-1/2})^2 + (1 - e^{-1/2})(e^{-1/2})^2\\
&\quad= 3e^{-1}(1 - e^{-1/2}).
\end{aligned}$$

6. (You may want to draw a tree diagram to help interpret the solution.) Let A be the event that a randomly selected Toyota vehicle is recalled. Let B_1 be the event that a randomly selected Toyota vehicle is a car, let B_2 be the event that a randomly selected Toyota vehicle is a truck, and let B_3 be the event that a randomly selected Toyota vehicle is a van. We are told that $\mathbf{P}\{B_1\} = 0.65$, $\mathbf{P}\{B_2\} = 0.20$, and $\mathbf{P}\{B_3\} = 0.15$. We are also told that $\mathbf{P}\{A \,|\, B_1\} = 0.10$, $\mathbf{P}\{A \,|\, B_2\} = 0.08$, and $\mathbf{P}\{A \,|\, B_3\} = 0.12$. We want to determine $\mathbf{P}\{B_2 \,|\, A\}$. Using Bayes' Rule we find

$$\begin{aligned}
\mathbf{P}\{B_2 \,|\, A\} = \frac{\mathbf{P}\{A \,|\, B_2\}\,\mathbf{P}\{B_2\}}{\mathbf{P}\{A\}} &= \frac{\mathbf{P}\{A \,|\, B_2\}\,\mathbf{P}\{B_2\}}{\mathbf{P}\{A \,|\, B_1\}\,\mathbf{P}\{B_1\} + \mathbf{P}\{A \,|\, B_2\}\,\mathbf{P}\{B_2\} + \mathbf{P}\{A \,|\, B_3\}\,\mathbf{P}\{B_3\}}\\
&= \frac{(0.08)(0.20)}{(0.10)(0.65) + (0.08)(0.20) + (0.12)(0.15)}\\
&= \frac{16}{99} \doteq 0.161616.
\end{aligned}$$