Math 302.102 Fall 2010
Solutions to Assignment #2

**1.** Let $R_j$, $j = 1, 2, \ldots, 50$, be the event that the $j$th box of Raisin Bran contains 2 scoops of raisins. We are told that $\mathbf{P}\{R_j\} = 0.93$ for each $j$ and that the events $R_1, R_2, \ldots, R_{50}$ are independent.

**(a)** The probability that all of Danny's 50 packages contain two scoops of raisins is

$$\mathbf{P}\{R_1 \cap R_2 \cap \cdots \cap R_{50}\} = \mathbf{P}\{R_1\}\mathbf{P}\{R_2\} \cdots \mathbf{P}\{R_{50}\} = (0.93)^{50}.$$

**(b)** Notice that

$$\{\text{at least one of Danny's 50 packages does not contain two scoops}\}^c$$
$$= \{\text{none of Danny's 50 packages do not contain two scoops}\}$$
$$= \{\text{all of Danny's 50 packages contain two scoops}\}$$

This implies that

$$\mathbf{P}\{\text{at least one of Danny's 50 packages does not contain two scoops}\}$$
$$= 1 - \mathbf{P}\{\text{all of Danny's 50 packages contain two scoops}\}$$
$$= 1 - (0.93)^{50}.$$

Symbolically, the event $\{\text{at least one of Danny's 50 packages does not contain two scoops}\}$ can be written as $R_1^c \cup R_2^c \cup \cdots \cup R_{50}^c$ and so

$$\mathbf{P}\{R_1^c \cup R_2^c \cup \cdots \cup R_{50}^c\} = 1 - \mathbf{P}\{(R_1^c \cup R_2^c \cup \cdots \cup R_{50}^c)^c\} = 1 - \mathbf{P}\{R_1 \cap R_2 \cap \cdots \cap R_{50}\}$$
$$= 1 - (0.93)^{50}$$

using the fact that $(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c = A \cap B$. (This is sometimes called De Morgan's law.)

**(c)** Notice that there are several ways in which exactly one package that does not contain two scoops of raisins could happen. It might be that only the first box does not contain two scoops, or it might be that only the second box does not contain two scoops, etc. Thus,

$\mathbf{P}\{\text{exactly one of Danny's 50 packages does not contain two scoops}\}$
$= \mathbf{P}\{(R_1^c \cap R_2 \cap \cdots \cap R_{50}) \cup (R_1 \cap R_2^c \cap R_3 \cap \cdots \cap R_{50}) \cup \cdots \cup (R_1 \cap R_2 \cap \cdots \cap R_{50}^c)\}$
$= \mathbf{P}\{R_1^c \cap R_2 \cap \cdots \cap R_{50}\} + \mathbf{P}\{R_1 \cap R_2^c \cap R_3 \cap \cdots \cap R_{50}\} + \cdots + \mathbf{P}\{R_1 \cap R_2 \cap \cdots \cap R_{50}^c\}$
$= (0.07)(0.93) \cdots (0.93) + (0.93)(0.07)(0.93) \cdots (0.93) + \cdots + (0.93) \cdots (0.93)(0.07)$
$= (0.07)(0.93)^{49} + (0.07)(0.93)^{49} + \cdots + (0.07)(0.93)^{49}$
$= 50(0.07)(0.93)^{49}.$

**2.** **(a)** Let $D_j$ be the event that the $j$th song is by Snoop Dogg. We are interested in computing $\mathbf{P}\{D_1 \cap D_2 \cap D_3\}$. The events $D_1$, $D_2$, and $D_3$ are NOT independent since we are not allowed to repeat a song in the playlist. Intuitively, there is a $3/20$ chance that the first song is by Snoop Dogg. Given that the first song is by Snoop Dogg, there is a $2/19$ chance that the second song will be by Snoop Dogg, and finally, given that the first two songs are by Snoop Dogg, the probability that the third will also be by him is $1/18$. Thus, the required probability is

$$\mathbf{P}\{D_1 \cap D_2 \cap D_3\} = \mathbf{P}\{D_1\}\mathbf{P}\{D_2 \,|\, D_1\}\mathbf{P}\{D_3 \,|\, D_1 \cap D_2\} = \frac{3}{20} \cdot \frac{2}{19} \cdot \frac{1}{18} = \frac{1}{1140}.$$

**(b)** This problem is solved exactly the same way as **(a)**. Thus, if $U_j$ is the event that the $j$th song is by U2, then the required probability is

$$\mathbf{P}\{U_1 \cap U_2 \cap U_3\} = \mathbf{P}\{U_1\}\mathbf{P}\{U_2 \,|\, U_1\}\mathbf{P}\{U_3 \,|\, U_1 \cap U_2\} = \frac{6}{20} \cdot \frac{5}{19} \cdot \frac{4}{18} = \frac{1}{57}.$$

**(c)** In order to solve this problem we will list all of the different ways that the first three songs can be by different artists. To begin, we select the three different artists (in any order). The possibilities are

$$\{\text{U2, Pink Floyd, Snoop Dogg}\}, \quad \{\text{U2, Pink Floyd, Sarah McLachlan}\},$$

$$\{\text{U2, Snoop Dogg, Sarah McLachlan}\}, \quad \{\text{Pink Floyd, Snoop Dogg, Sarah McLachlan}\}.$$

Thus, if $U$, $D$, $P$, and $S$ are the events that one of the first three songs are by U2, Snoop Dogg, Pink Floyd, and Sarah McLachlan, respectively, then

$$\mathbf{P}\{\text{1st three songs are by different artists}\}$$
$$= \mathbf{P}\{(U \cap P \cap D) \cup (U \cap P \cap S) \cup (U \cap D \cap S) \cup (P \cap D \cap S)\}$$
$$= \mathbf{P}\{U \cap P \cap D\} + \mathbf{P}\{U \cap P \cap S\} + \mathbf{P}\{U \cap D \cap S\} + \mathbf{P}\{P \cap D \cap S\}.$$

However, within each group, there are six different orderings. That is, if the group is $\{a, b, c\}$, then the orderings are $abc$, $acb$, $bac$, $bca$, $cab$, $cba$. This means that in order to compute $\mathbf{P}\{U \cap P \cap D\}$, we must consider the six different ways that the first three songs can be by U2, Pink Floyd, and Snoop Dogg. Thus, if $U_j$, $P_j$, and $D_j$ represent the events that the $j$th song is by U2, Pink Floyd, and Snoop Dogg, respectively, then

$$\mathbf{P}\{U \cap P \cap D\}$$
$$= \mathbf{P}\{(U_1 \cap P_2 \cap D_3) \cup (U_1 \cap D_2 \cap P_3) \cup (P_1 \cap U_2 \cap D_3) \cup (P_1 \cap D_2 \cap U_3)$$
$$\qquad \cup (D_1 \cap P_2 \cap U_3) \cup (D_1 \cap U_2 \cap P_3)\}$$
$$= \mathbf{P}\{U_1 \cap P_2 \cap D_3\} + \mathbf{P}\{U_1 \cap D_2 \cap P_3\} + \mathbf{P}\{P_1 \cap U_2 \cap D_3\} + \mathbf{P}\{P_1 \cap D_2 \cap U_3\}$$
$$\qquad + \mathbf{P}\{D_1 \cap P_2 \cap U_3\} + \mathbf{P}\{D_1 \cap U_2 \cap P_3\}$$
$$= \frac{6}{20} \cdot \frac{4}{19} \cdot \frac{3}{18} + \frac{6}{20} \cdot \frac{3}{19} \cdot \frac{4}{18} + \frac{4}{20} \cdot \frac{6}{19} \cdot \frac{3}{18} + \frac{4}{20} \cdot \frac{3}{19} \cdot \frac{6}{18} + \frac{3}{20} \cdot \frac{4}{19} \cdot \frac{6}{18} + \frac{3}{20} \cdot \frac{6}{19} \cdot \frac{4}{18}$$
$$= 6 \cdot \frac{3 \cdot 4 \cdot 6}{20 \cdot 19 \cdot 18}.$$

Similarly,

$$\mathbf{P}\{U \cap P \cap S\} = 6 \cdot \frac{6 \cdot 4 \cdot 7}{20 \cdot 19 \cdot 18}, \quad \mathbf{P}\{U \cap D \cap S\} = 6 \cdot \frac{6 \cdot 3 \cdot 7}{20 \cdot 19 \cdot 18}, \quad \mathbf{P}\{P \cap D \cap S\} = 6 \cdot \frac{4 \cdot 3 \cdot 7}{20 \cdot 19 \cdot 18}.$$

Therefore, we conclude that

$$\mathbf{P}\{\text{1st three songs are by different artists}\}$$
$$= 6 \cdot \frac{3 \cdot 4 \cdot 6}{20 \cdot 19 \cdot 18} + 6 \cdot \frac{6 \cdot 4 \cdot 7}{20 \cdot 19 \cdot 18} + 6 \cdot \frac{6 \cdot 3 \cdot 7}{20 \cdot 19 \cdot 18} + 6 \cdot \frac{4 \cdot 3 \cdot 7}{20 \cdot 19 \cdot 18}$$
$$= \frac{15}{38}.$$

**3.** We want to compute $\mathbf{P}\{\text{Elvis was an identical twin} \mid \text{Elvis had a twin brother}\}$. Using Bayes' rule we find

$$\mathbf{P}\{\text{Elvis was an identical twin}|\text{Elvis had a twin brother}\}$$
$$= \frac{\mathbf{P}\{\text{Elvis had a twin brother}|\text{Elvis was an identical twin}\} \cdot \mathbf{P}\{\text{Elvis was an identical twin}\}}{\mathbf{P}\{\text{Elvis had a twin brother}\}}.$$

We are told that

$$\mathbf{P}\{\text{Elvis was an identical twin}\} = \frac{1}{300}.$$

Furthermore, it follows immediately that

$$\mathbf{P}\{\text{Elvis had a twin brother} \mid \text{Elvis was an identical twin}\} = 1.$$

Therefore, we must calculate $\mathbf{P}\{\text{Elvis had a twin brother}\}$ using the Law of Total Probability. Thus,

$$\mathbf{P}\{\text{Elvis had a twin brother}\}$$
$$= \mathbf{P}\{\text{Elvis had a twin brother} \mid \text{Elvis was an identical twin}\} \cdot \mathbf{P}\{\text{Elvis was an identical twin}\}$$
$$+ \mathbf{P}\{\text{Elvis had a twin brother} \mid \text{Elvis was a fraternal twin}\} \cdot \mathbf{P}\{\text{Elvis was a fraternal twin}\}$$
$$+ \mathbf{P}\{\text{Elvis had a twin brother} \mid \text{Elvis was NOT a twin}\} \cdot \mathbf{P}\{\text{Elvis was NOT a twin}\}$$
$$= 1 \cdot \frac{1}{300} + \frac{1}{2} \cdot \frac{1}{125} + 0$$

so that

$$\mathbf{P}\{\text{Elvis was an identical twin} \mid \text{Elvis had a twin brother}\} = \frac{1 \cdot \frac{1}{300}}{1 \cdot \frac{1}{300} + \frac{1}{2} \cdot \frac{1}{125}} = \frac{5}{11}.$$

**4.** The following tree diagram shows the possible values of the stock on days 3, 4, and 5, and the corresponding probabilities. Note that tomorrow the stock has a 60% chance of doing the same thing (either increasing or decreasing) as today.
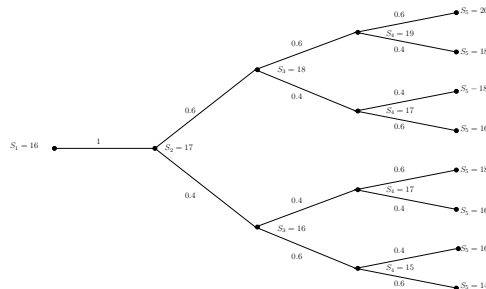


We can now answer both **(a)** and **(b)** by following the appropriate branches through the tree.

**(a)** We know the values of the stock on days 1 and 2. Therefore, in order to determine the probability that the stock price is $17 on day 4, we need to use the Law of Total Probability and condition on the possible values of the stock on day 3. Thus,

$$
\begin{aligned}
\mathbf{P}\left\{S_4 = 17\right\} &= \mathbf{P}\left\{S_4 = 17 \cap S_3 = 16\right\} + \mathbf{P}\left\{S_4 = 17 \cap S_3 = 18\right\} \\
&= \mathbf{P}\left\{S_4 = 17 \,|\, S_3 = 16\right\} \mathbf{P}\left\{S_3 = 16\right\} + \mathbf{P}\left\{S_4 = 17 \,|\, S_3 = 18\right\} \mathbf{P}\left\{S_3 = 18\right\} \\
&= (0.4)(0.4) + (0.4)(0.6) \\
&= 0.4.
\end{aligned}
$$

**(b)** If we know that the price of the stock was $18 on day 5, then in order to determine the probability that the stock price was $17 on day 4, we need to consider what happened on day 3. Intuitively, we can just use the definition of conditional probability and notice that there are three paths that lead to $S_5 = 18$. Of those, there are two that have $S_4 = 17$. Thus,

$$
\mathbf{P}\left\{S_4 = 17 \,|\, S_5 = 18\right\} = \frac{(0.4)(0.4)(0.6) + (0.6)(0.4)(0.4)}{(0.4)(0.4)(0.6) + (0.6)(0.4)(0.4) + (0.6)(0.6)(0.4)} = \frac{4}{7}.
$$

Equivalently, we find

$$
\mathbf{P}\left\{S_4 = 17 \,|\, S_5 = 18\right\} = \frac{\mathbf{P}\left\{S_4 = 17 \cap S_5 = 18\right\}}{\mathbf{P}\left\{S_5 = 18\right\}}.
$$

We now use the Law of Total Probability in both the numerator and denominator. That is,

$$
\begin{aligned}
\mathbf{P}\left\{S_5 = 18\right\} &= \mathbf{P}\left\{S_5 = 18 \cap S_4 = 17\right\} + \mathbf{P}\left\{S_5 = 18 \cap S_4 = 19\right\} \\
&= \mathbf{P}\left\{S_5 = 18 \cap S_4 = 17 \cap S_3 = 16\right\} + \mathbf{P}\left\{S_5 = 18 \cap S_4 = 17 \cap S_3 = 18\right\} \\
&\quad + \mathbf{P}\left\{S_5 = 18 \cap S_4 = 19 \cap S_3 = 18\right\} \\
&= (0.4)(0.4)(0.6) + (0.6)(0.4)(0.4) + (0.6)(0.6)(0.4)
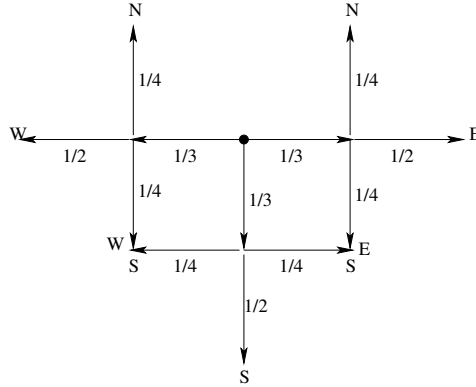\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{P}\left\{S_4 = 17 \cap S_5 = 18\right\} &= \mathbf{P}\left\{S_5 = 18 \cap S_4 = 17 \cap S_3 = 16\right\} + \mathbf{P}\left\{S_5 = 18 \cap S_4 = 17 \cap S_3 = 18\right\} \\
&= (0.4)(0.4)(0.6) + (0.6)(0.4)(0.4)
\end{aligned}
$$

giving

$$
\mathbf{P}\left\{S_4 = 17 \,|\, S_5 = 18\right\} = \frac{(0.4)(0.4)(0.6) + (0.6)(0.4)(0.4)}{(0.4)(0.4)(0.6) + (0.6)(0.4)(0.4) + (0.6)(0.6)(0.4)} = \frac{4}{7}
$$

as above. Note that you could also use Bayes' Rule, but in order to do so you must account for what happens on day 3. Hence, you will be lead to precisely the same calculation.

**5.** In order to solve this problem, we make a *map* (i.e., a tree diagram) of the possible routes that Michael could take. Note that on the first movement, there is probability $1/3$ that he moves east, south, or west, respectively. On the second movement, he changes his strategy by flipping coins as indicated in the problem.

Thus, we see that there are two possible routes that leave Michael facing west: (i) west first then straight, or (ii) south first then right. The required probability is therefore

$\mathbf{P}\{$Michael faces west after second movement$\}$

$= \mathbf{P}\{$first west, second straight OR first south, second right$\}$

$= \mathbf{P}\{$first west, second straight$\} + \mathbf{P}\{$first south, second right$\}$

$= \mathbf{P}\{$second straight $\mid$ first west$\}\,\mathbf{P}\{$first west$\} + \mathbf{P}\{$second right $\mid$ first south$\}\,\mathbf{P}\{$first south$\}$

$= \mathbf{P}\{$heads$\} \cdot \dfrac{1}{3} + \mathbf{P}\{$tails, then heads$\} \cdot \dfrac{1}{3}$

$= \dfrac{1}{2} \cdot \dfrac{1}{3} + \dfrac{1}{4} \cdot \dfrac{1}{3}$

$= \dfrac{1}{4}.$


**6.**  In order for the batter to safely reach base, the batter must safely reach base on the first full count pitch, or the batter must hit a foul ball and then reach base on the second full count pitch, or the batter must hit two foul balls and then reach base safely on the third full count pitch, etc. Let $F_j$ denote the event that the batter hits a foul ball on the $j$th pitch facing a full count, and let $S_j$ denote the event that the batter safely reaches base on the $j$th pitch facing a full count. We are told that $\mathbf{P}\{F_j\} = 0.15$ and $\mathbf{P}\{S_j\} = 0.4$ for all $j$. Therefore,

$\mathbf{P}\{$eventually safely reach base$\}$

$\quad = \mathbf{P}\{S_1 \cup (F_1 \cap S_2) \cup (F_1 \cap F_2 \cap S_3) \cup (F_1 \cap F_2 \cap F_3 \cap S_4) \cup \cdots\}$

$\quad = \mathbf{P}\{S_1\} + \mathbf{P}\{F_1 \cap S_2\} + \mathbf{P}\{F_1 \cap F_2 \cap S_3\} + \mathbf{P}\{F_1 \cap F_2 \cap F_3 \cap S_4\} + \cdots$

$\quad = \mathbf{P}\{S_1\} + \mathbf{P}\{F_1\}\mathbf{P}\{S_2\} + \mathbf{P}\{F_1\}\mathbf{P}\{F_2\}\mathbf{P}\{S_3\} + \mathbf{P}\{F_1\}\mathbf{P}\{F_2\}\mathbf{P}\{F_3\}\mathbf{P}\{S_4\} + \cdots$

$\quad = (0.4) + (0.15)(0.4) + (0.15)^2(0.4) + (0.15)^3(0.4) + \cdots$

$\quad = (0.4)[1 + (0.15) + (0.15)^2 + (0.15)^3 + \cdots]$

$\quad = (0.4) \cdot \dfrac{1}{1 - 0.15}$

$\quad = \dfrac{40}{85}$

$\quad = \dfrac{8}{17}.$