Math 302.102 Fall 2010
Solutions to Assignment #1

1. **(a)** If $S = \{a, b, c, d\}$ is the sample space consisting of 4 outcomes, then there are $2^4 = 16$ possible events. They can be enumerated by listing all events containing 4 elements, namely $\{a, b, c, d\}$, all events containing 3 elements, namely $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$, all events containing 2 elements, namely $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$, all events containing 1 element, namely $\{a\}, \{b\}, \{c\}, \{d\}$, and all events containing 0 elements, namely $\emptyset$. That is,

$$\mathcal{F} = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},$$
$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\Big\}.$$

(b) If all events are equally likely, then the probability of any event is simply the number of outcomes in that event divided by 4. That is,

$$\mathbf{P}\{A\} = \frac{\#(A)}{4}$$

for any $A \in \mathcal{F}$ where $\#(A)$ denotes the number of elements in A.

2. In order to solve this problem, we need to consider all possible scenarios where player 1 (your opponent) chooses a coin and then player 2 (you) chooses a coin. There are six separate scenarios that we will consider in turn.

(i) Suppose that player 1 chooses A and player 2 chooses B. Then

$$\mathbf{P}\{\text{player 2 wins}\} = \mathbf{P}\{\text{coin } A \text{ shows tails}\} = \frac{2}{5}.$$

(ii) Suppose that player 1 chooses A and player 2 chooses C. Then

$$\begin{aligned}\mathbf{P}\{\text{player 2 wins}\} &= \mathbf{P}\{(\text{coin } C \text{ shows tails}) \text{ or } (\text{coin } A \text{ shows tails and } C \text{ shows heads})\} \\ &= \mathbf{P}\{\text{coin } C \text{ shows tails}\} + \mathbf{P}\{\text{coin } A \text{ shows tails and } C \text{ shows heads}\} \\ &= \frac{2}{5} + \frac{2}{5} \cdot \frac{3}{5} \\ &= \frac{16}{25}.\end{aligned}$$

(iii) Suppose that player 1 chooses B and player 2 chooses A. Then

$$\mathbf{P}\{\text{player 2 wins}\} = \mathbf{P}\{\text{coin } A \text{ shows heads}\} = \frac{3}{5}.$$

(iv) Suppose that player 1 chooses B and player 2 chooses C. Then

$$\mathbf{P}\{\text{player 2 wins}\} = \mathbf{P}\{\text{coin } C \text{ shows tails}\} = \frac{2}{5}.$$

(v) Suppose that player 1 chooses C and player 2 chooses A. Then

$$\begin{aligned}\mathbf{P}\{\text{player 2 wins}\} &= \mathbf{P}\{\text{coin } C \text{ shows heads and } A \text{ shows heads}\}\\ &= \frac{3}{5}\cdot\frac{3}{5}\\ &= \frac{9}{25}.\end{aligned}$$

(vi) Suppose that player 1 chooses C and player 2 chooses B. Then

$$\mathbf{P}\{\text{player 2 wins}\} = \mathbf{P}\{\text{coin } A \text{ shows heads}\} = \frac{3}{5}.$$

Hence, no matter which coin player 1 chooses, then there is a coin that player 2 can choose which gives player 2 a probability of winning greater than 1/2. That is, if player 1 chooses coin A, then player 2 can choose coin C. If player 1 chooses coin B, then player 2 can choose coin A. Finally, if player 1 chooses coin C, then player 2 can choose coin B. Thus, it is better to be the second to choose a coin.

3. **(a)** Let A_i denote the event that a bell appears on wheel i so that

$$\mathbf{P}\{A_1\} = \frac{1}{20}, \quad \mathbf{P}\{A_2\} = \frac{9}{20}, \quad \mathbf{P}\{A_3\} = \frac{1}{20}.$$

The wheels are assumed to be independent so that

$$\mathbf{P}\{\text{jackpot}\} = \mathbf{P}\{A_1 \cap A_2 \cap A_3\} = \mathbf{P}\{A_1\}\mathbf{P}\{A_2\}\mathbf{P}\{A_3\} = \frac{1}{20}\cdot\frac{9}{20}\cdot\frac{1}{20} = \frac{9}{8000}.$$

(b) Notice that there are three mutually exclusive ways of obtaining exactly 2 bells, namely

$$A_1^c \cap A_2 \cap A_3, \quad A_1 \cap A_2^c \cap A_3, \quad A_1 \cap A_2 \cap A_3^c.$$

Therefore, we find

$$\begin{aligned}\mathbf{P}\{\text{exactly 2 bells}\} &= \mathbf{P}\{A_1^c \cap A_2 \cap A_3 \text{ or } A_1 \cap A_2^c \cap A_3 \text{ or } A_1 \cap A_2 \cap A_3^c\}\\ &= \mathbf{P}\{A_1^c \cap A_2 \cap A_3\} + \mathbf{P}\{A_1 \cap A_2^c \cap A_3\} + \mathbf{P}\{A_1 \cap A_2 \cap A_3^c\}\\ &= \mathbf{P}\{A_1^c\}\mathbf{P}\{A_2\}\mathbf{P}\{A_3\} + \mathbf{P}\{A_1\}\mathbf{P}\{A_2^c\}\mathbf{P}\{A_3\} + \mathbf{P}\{A_1\}\mathbf{P}\{A_2\}\mathbf{P}\{A_3^c\}\\ &= \frac{19}{20}\cdot\frac{9}{20}\cdot\frac{1}{20} + \frac{1}{20}\cdot\frac{11}{20}\cdot\frac{1}{20} + \frac{1}{20}\cdot\frac{9}{20}\cdot\frac{19}{20}\\ &= \frac{353}{8000}.\end{aligned}$$

(c) Suppose instead that there are three bells on the left, one in the middle, and three on the right. Let B_i denote the event that a bell appears on wheel i so that

$$\mathbf{P}\{B_1\} = \frac{3}{20}, \quad \mathbf{P}\{B_2\} = \frac{1}{20}, \quad \mathbf{P}\{B_3\} = \frac{3}{20}.$$

The wheels are assumed to be independent so that

$$\mathbf{P}\{\text{jackpot}\} = \mathbf{P}\{B_1 \cap B_2 \cap B_3\} = \mathbf{P}\{B_1\}\mathbf{P}\{B_2\}\mathbf{P}\{B_3\} = \frac{3}{20}\cdot\frac{1}{20}\cdot\frac{3}{20} = \frac{9}{8000}.$$

Furthermore, we find

$$\begin{aligned}\mathbf{P}\{\text{exactly 2 bells}\} &= \mathbf{P}\{B_1^c \cap B_2 \cap B_3 \text{ or } B_1 \cap B_2^c \cap B_3 \text{ or } B_1 \cap B_2 \cap B_3^c\} \\ &= \mathbf{P}\{B_1^c \cap B_2 \cap B_3\} + \mathbf{P}\{B_1 \cap B_2^c \cap B_3\} + \mathbf{P}\{B_1 \cap B_2 \cap B_3^c\} \\ &= \mathbf{P}\{B_1^c\}\mathbf{P}\{B_2\}\mathbf{P}\{B_3\} + \mathbf{P}\{B_1\}\mathbf{P}\{B_2^c\}\mathbf{P}\{B_3\} + \mathbf{P}\{B_1\}\mathbf{P}\{B_2\}\mathbf{P}\{B_3^c\} \\ &= \frac{17}{20}\cdot\frac{1}{20}\cdot\frac{3}{20} + \frac{3}{20}\cdot\frac{19}{20}\cdot\frac{3}{20} + \frac{3}{20}\cdot\frac{1}{20}\cdot\frac{17}{20} \\ &= \frac{273}{8000}.\end{aligned}$$

The casino might prefer the 1-9-1 machine to the 3-1-3 machine for purely psychological reasons. Even though both machines have the same probability of delivering a jackpot (and the casino only pays out for jackpots), the first 1-9-1 machine might give players the impression of having more bells than the 3-1-3 machine. In other words, on almost half the plays of the 1-9-1 machine (actually on 9/20th of the plays), there will be a bell on the middle wheel. Thus, on most plays, there will be bells appearing on the 1-9-1 machine. This might entice players into thinking they are close to hitting a jackpot since they are seeing so many bells.

4. The key to solving this problem is to realize that once a heads appears, it is not possible for the string TTT to appear without first the string HTT appearing. In other words, although HTTT does contain the string TTT, the string HTT has already appeared. Thus, the only way for TTT to ever appear before HTT is if TTT appears on the first three tosses. Thus,

$$\mathbf{P}\{\text{player B wins}\} = 1 - \mathbf{P}\{\text{player A wins}\} = 1 - \mathbf{P}\{\text{TTT on first three tosses}\} = 1 - \frac{1}{8} = \frac{7}{8}.$$

5. There are three distinct paths that current can take in order to flow from left-to-right. The first, call it *top*, is through switches 1 and 3; the second, call it *middle*, is through switches 2 and 3; and the third, call it *bottom*, is through switch 4. Therefore, if A, B, and C, represent the events that current flows left-to-right through the top path, middle path, and bottom path, respectively, then in order for current to flow in this circuit, it must be the case that at least one of the three paths is closed. In set notation, this is the event $A \cup B \cup C$. In order to compute $\mathbf{P}\{A \cup B \cup C\}$ we consider it in two pieces. First, we find that

$$\mathbf{P}\{A \cup B\} = \mathbf{P}\{A\} + \mathbf{P}\{B\} - \mathbf{P}\{A \cap B\}.$$

If D is the event $D = A \cup B$, then we can write $A \cup B \cup C = D \cup C$ so that

$$\begin{aligned}\mathbf{P}\{A \cup B \cup C\} = \mathbf{P}\{D \cup C\} &= \mathbf{P}\{D\} + \mathbf{P}\{C\} - \mathbf{P}\{D \cap C\} \\ &= \mathbf{P}\{A \cup B\} + \mathbf{P}\{C\} - \mathbf{P}\{(A \cup B) \cap C\} \\ &= \mathbf{P}\{A\} + \mathbf{P}\{B\} - \mathbf{P}\{A \cap B\} + \mathbf{P}\{C\} - \mathbf{P}\{(A \cup B) \cap C\}.\end{aligned}$$

However, notice that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, and so

$$\begin{aligned}\mathbf{P}\{(A \cup B) \cap C\} &= \mathbf{P}\{(A \cap C) \cup (B \cap C)\} \\ &= \mathbf{P}\{A \cap C\} + \mathbf{P}\{B \cap C\} - \mathbf{P}\{A \cap C \cap B \cap C\} \\ &= \mathbf{P}\{A \cap C\} + \mathbf{P}\{B \cap C\} - \mathbf{P}\{A \cap B \cap C\}.\end{aligned}$$

Finally, if we substitute this into the expression for $\mathbf{P}\{(A \cup B) \cap C\}$ above, we conclude

$$\begin{aligned} &\mathbf{P}\{A \cup B \cup C\} \\ &\qquad = \mathbf{P}\{A\} + \mathbf{P}\{B\} + \mathbf{P}\{C\} - \mathbf{P}\{A \cap B\} - \mathbf{P}\{A \cap C\} - \mathbf{P}\{B \cap C\} + \mathbf{P}\{A \cap B \cap C\}. \end{aligned}$$

The fact that the switches function independently implies that

$$\begin{aligned} \mathbf{P}\{A\} = \mathbf{P}\{\text{current flows along top path}\} &= \mathbf{P}\{\text{switch 1 is closed and switch 3 is closed}\} \\ &= \mathbf{P}\{\text{switch 1 is closed}\}\,\mathbf{P}\{\text{switch 3 is closed}\} \\ &= p_1 p_3, \end{aligned}$$

and similarly

$$\mathbf{P}\{B\} = p_2 p_3, \quad \text{and} \quad \mathbf{P}\{C\} = p_4.$$

Furthermore, independence of the switches implies that

$$\begin{aligned} \mathbf{P}\{A \cap B\} &= \mathbf{P}\{\text{current flows along top path and along middle path}\} \\ &= \mathbf{P}\{\text{switch 1 is closed and switch 2 is closed and switch 3 is closed}\} \\ &= \mathbf{P}\{\text{switch 1 is closed}\}\,\mathbf{P}\{\text{switch 2 is closed}\}\,\mathbf{P}\{\text{switch 3 is closed}\} \\ &= p_1 p_2 p_3 \end{aligned}$$

and similarly

$$\mathbf{P}\{A \cap C\} = p_1 p_3 p_4, \quad \mathbf{P}\{B \cap C\} = p_2 p_3 p_4, \quad \mathbf{P}\{A \cap B \cap C\} = p_1 p_2 p_3 p_4.$$

Hence, putting everything together gives

$$\mathbf{P}\{\text{current flows}\} = p_1 p_3 + p_2 p_3 + p_4 - p_1 p_2 p_3 - p_1 p_3 p_4 - p_2 p_3 p_4 + p_1 p_2 p_3 p_4.$$

It is not important, but his can actually be simplified slightly. That is,

$$\begin{aligned} &p_1 p_3 + p_2 p_3 + p_4 - p_1 p_2 p_3 - p_1 p_3 p_4 - p_2 p_3 p_4 + p_1 p_2 p_3 p_4 \\ &\qquad = p_1 p_3 - p_1 p_2 p_3 + p_2 p_3 - p_2 p_3 p_4 - p_1 p_3 p_4 + p_1 p_2 p_3 p_4 + p_4 \\ &\qquad = p_1 p_3 (1 - p_2) + p_2 p_3 (1 - p_4) - p_1 p_3 p_4 (1 - p_2) + p_4 \\ &\qquad = p_1 q_2 p_3 + p_2 p_3 q_4 - p_1 q_2 p_3 p_4 + p_4 \\ &\qquad = p_1 q_2 p_3 - p_1 q_2 p_3 p_4 + p_2 p_3 q_4 + p_4 \\ &\qquad = p_1 q_2 p_3 (1 - p_4) + p_2 p_3 q_4 + p_4 \\ &\qquad = p_1 q_2 p_3 q_4 + p_2 p_3 q_4 + p_4. \end{aligned}$$

In fact, this suggests an alternative solution to the problem. Notice that

$$p_1 q_2 p_3 q_4 + p_2 p_3 q_4 + p_4 = \mathbf{P}\{A \cap B^c \cap C^c\} + \mathbf{P}\{B \cap C^c\} + \mathbf{P}\{C\}.$$

From a Venn diagram, notice that we can write $A \cup B \cup C$ as a disjoint union as

$$A \cup B \cup C = (A \cap B^c \cap C^c) \cup (B \cap C^c) \cup C.$$

This gives

$$\mathbf{P}\{A \cup B \cup C\} = \mathbf{P}\{A \cap B^c \cap C^c\} + \mathbf{P}\{B \cap C^c\} + \mathbf{P}\{C\} = p_1 q_2 p_3 q_4 + p_2 p_3 q_4 + p_4.$$