Math 302.102 Fall 2010
Assignment #8

This assignment is due at the beginning of class on Monday, November 29, 2010.

1. A box contains three white balls and four red balls. Suppose that 100 balls are drawn from this box at random with replacement. Write down an exact binomial expression for the probability that at least 50 balls drawn are red. In other words, if X denotes the number of red balls drawn, determine $\mathbf{P}\{X \geq 50\}$. Use a suitable normal approximation to estimate this probability. (Even if you can determine the binomial probability exactly, I still want you to do the normal approximation.)

2. Suppose that you play the following game. You pay \$4 and roll a single standard six-sided die. You win \$$X$ if an X appears on your roll. Thus, your net winnings (or net gain if you prefer) is $X - 4$. Suppose that you then play this game 100 times. Use a suitable normal approximation to answer the following questions.

(a) What is the probability that you will lose exactly \$50?

(b) What is the probability that you will win? (In other words, what is the probability that your total net winnings after 100 plays will be strictly positive?)

3. You roll a fair die once and record the number observed as X. I then roll the same die repeatedly until I obtain a number which is greater than or equal to your number X. Let Y denote the number of rolls that I have to make.

(a) Determine $\mathbf{P}\{X = 1, Y = k\}$ for all $k = 1, 2, 3, \ldots$.

(b) Determine $\mathbf{P}\{X = 2, Y = k\}$ for all $k = 1, 2, 3, \ldots$.

(c) Find the joint distribution table for X and Y. That is, fill in a table with the values of $\mathbf{P}\{X = i, Y = k\}$ for $i = 1, 2, 3, 4, 5, 6$ and $k = 1, 2, 3, \ldots$. You have already done $i = 1$ and $i = 2$ in the previous two parts.

(d) Determine $\mathbb{E}(Y)$. It is easiest if you find the marginal for Y using the law of total probability and the joint distribution table; that is,

$$\mathbf{P}\{Y = k\} = \sum_{i=1}^{6} \mathbf{P}\{X = i, Y = k\}.$$

Now write

$$\mathbb{E}(Y) = \sum_{k=1}^{\infty} k\mathbf{P}\{Y = k\} = \sum_{k=1}^{\infty} k \sum_{i=1}^{6} \mathbf{P}\{X = i, Y = k\} = \sum_{k=1}^{\infty} \sum_{i=1}^{6} k\mathbf{P}\{X = i, Y = k\}$$

and then switch the order of summation so that

$$\mathbb{E}(Y) = \sum_{i=1}^{6} \sum_{k=1}^{\infty} k\mathbf{P}\{X = i, Y = k\}.$$

4. Suppose that X and Y are independent random variables such that X has a normal distribution with mean 3 and variance 2, and that Y has an exponential distribution with mean 1/2. Determine $\mathbb{E}(X^2 e^Y)$.

5. Let (X, Y) be uniformly distributed on the triangle $\{(x, y) : 0 < x < y < 1\}$.

(a) Determine the density function $f(x, y) = f_{X,Y}(x, y)$ for (X, Y).

(b) Compute $\mathbf{P}\left\{X < Y^2\right\}$. Note that X and Y are NOT independent. Therefore, the law of total probability does not apply. Instead, you find the probability by integrating the density function. That is,

$$\mathbf{P}\left\{(X, Y) \in A\right\} = \iint_A f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \iint_A f(x, y)\, \mathrm{d}y\, \mathrm{d}x.$$

You will need to decide which order of integration is easiest for you.

(c) Determine $f_X(x)$, the marginal density for X.

(d) Suppose that 100 points are chosen independent and uniformly at random from the triangle. What is the probability that the average of their x-coordinate is greater than 7/18?

6. There are n kernels of popcorn in a pan. The number of these kernels that pop is a random variable with expected value 63 and variance 6.3. What is the distribution of X? *Assume that whether or not an individual kernel pops is independent of the behaviour of the other kernels.* Note that you must find n.

7. An opinion poll takes a random sample of 1350 voters in a large city. Suppose that 40% of the population support Party A, while 0.2% of the population support Party B. Find numerical approximations (using different distributions) for

(a) the probability that more than 544 of the people in the sample support Party A, and

(b) the probability that at most 2 of the people in the sample support Party B.