Stat 862 (Winter 2007)
Pólya's Theorem

Let $p_{2n} := p_{2n}(0,0) := P\{X_{2n} = 0 | X_0 = 0\}$ and suppose that $d = 2$. To return to the origin, the walker must take

- the same number of steps left as right, AND
- the same number of steps up as down.

Therefore, every path that returns in $2n$ steps has probability $\left(\frac{1}{4}\right)^{2n}$ of occurring. The number of paths with k steps left, k steps right, $n-k$ steps up, $n-k$ steps down is

$$\binom{2n}{k, k, n-k, n-k} := \frac{(2n)!}{k!k!(n-k)!(n-k)!},$$

and so

$$\begin{aligned} p_{2n} = \left(\frac{1}{4}\right)^{2n} \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!} &= \left(\frac{1}{4}\right)^{2n} \sum_{k=0}^{n} \frac{(2n)!}{n!n!} \frac{n!n!}{k!k!(n-k)!(n-k)!} \\ &= \left(\frac{1}{4}\right)^{2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \end{aligned}$$

Note that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

and so

$$p_{2n} = \left(\frac{1}{2^{2n}} \binom{2n}{n}\right)^2.$$

Thus, using Stirling's formula,

$$\sum_{n} p_{2n} \sim \sum_{n} \frac{1}{\pi n} = \infty.$$

Notice that this is just the square of the one dimensional result! That is, simple random walk in two dimensions is recurrent.

Suppose now that $d = 3$. In order to return to the origin, the walker must take

- the same number of steps left as right, AND
- the same number of steps up as down, AND
- the same number of steps forward as backward.

Therefore, every path that returns to the origin in $2n$ steps has probability $\left(\frac{1}{6}\right)^{2n}$ of occurring. The number of paths with k steps left, k steps right, j steps up, j steps down, $n-k-j$ steps forward, $n-j-k$ steps backward is

$$\binom{2n}{k,k,j,j,n-k-j,n-k-j} := \frac{(2n)!}{k!k!j!j!(n-k-j)!(n-k-j)!}$$

and so

$$p_{2n} = \frac{1}{6^{2n}} \sum_{\substack{j,k \\ j+k\leq n}} \frac{(2n)!}{k!k!j!j!(n-j-k)!(n-j-k)!} = \frac{1}{2^{2n}}\binom{2n}{n} \sum_{\substack{j,k \\ j+k\leq n}} \left(\frac{1}{3^n}\frac{n!}{k!j!(n-k-j)!}\right)^2.$$

Now,

$$\begin{aligned}\frac{1}{3^n}\binom{n}{k,j,n-j-k} &= \frac{1}{3^n}\frac{n!}{k!j!(n-j-k)!} \\ &= \text{probability of placing } n \text{ balls in 3 boxes}\end{aligned}$$

This is maximized when k, j, $(n-k-j)$ are as close to $\frac{n}{3}$ as possible. Therefore,

$$p_{2n} \leq \frac{1}{2^{2n}}\binom{2n}{n}\left(\frac{1}{3^n}\frac{n!}{\left[\frac{n}{3}\right]!\left[\frac{n}{3}\right]!\left[\frac{n}{3}\right]!}\right)\underbrace{\left(\sum_{j,k}\frac{1}{3^n}\frac{n!}{k!j!(n-j-k)!}\right)}_{=1 \text{ since it is a distribution}}$$

and so

$$p_{2n} \leq \frac{1}{2^{2n}}\binom{2n}{n}\left(\frac{1}{3^n}\frac{n!}{\left(\left[\frac{n}{3}\right]!\right)^3}\right)$$

However, Stirling's formula implies $p_{2n} \leq \frac{K}{n^{3/2}}$ for some constant $K \in \mathbb{R}^+$, and so

$$\sum_n p_{2n} \leq K \sum_n \frac{1}{n^{3/2}} < \infty.$$

That is, simple random walk in three dimensions is transient.