Stat 862 (Winter 2007)
Optimal Stopping

Consider the game described in Section 4.1 for which

$$f(k) = \begin{cases} k, & \text{if } k = 1,2,3,4,5, \\ 0, & \text{otherwise.} \end{cases}$$

The transition matrix for the underlying Markov chain is

$$\mathbf{P} = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we set

$$u_1(x) = \begin{cases} f(x), & \text{if } x \text{ is an absorbing state,} \\ \max_{x \in S} f(x), & \text{if } x \text{ is not an absorbing state,} \end{cases}$$

and

$$u_n(x) = \max\{\mathbf{P}u_{n-1}(x), f(x)\}$$

for $n = 2, 3, \ldots$, then

$$v(x) = \lim_{n \to \infty} u_n(x).$$

Since the only absorbing state is $x = 6$, we find $u_1(6) = 0$, and since $\max_{x \in S} f(x) = 5$, we find $u_1(1) = u_1(2) = u_1(3) = u_1(4) = u_1(5) = 5$. That is,

$$u_1 = [5, 5, 5, 5, 5, 0].$$

We then calculate

$$\mathbf{P}u_1 = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 0 \end{bmatrix} = \left[\frac{25}{6}, \frac{25}{6}, \frac{25}{6}, \frac{25}{6}, \frac{25}{6}, 0\right]$$

and so $u_2(x) = \max\{\mathbf{P}u_1(x), f(x)\}$ implies

$$u_2 = \left[\frac{25}{6}, \frac{25}{6}, \frac{25}{6}, \frac{25}{6}, 5, 0\right].$$

Next we calculate

$$\mathbf{P}u_2 = \left[\frac{65}{18}, \frac{65}{18}, \frac{65}{18}, \frac{65}{18}, \frac{65}{18}, 0\right]$$

and so $u_3(x) = \max\{\mathbf{P}u_2(x), f(x)\}$ implies

$$u_3 = \left[\frac{65}{18}, \frac{65}{18}, \frac{65}{18}, 4, 5, 0\right].$$

Continuing in this fashion gives

$$u_4 = \left[\frac{119}{36}, \frac{119}{36}, \frac{119}{36}, 4, 5, 0\right],$$

$$u_5 = \left[\frac{227}{72}, \frac{227}{72}, \frac{227}{72}, 4, 5, 0\right],$$

$$u_6 = \left[\frac{443}{144}, \frac{443}{144}, \frac{443}{144}, 4, 5, 0\right],$$

$$u_7 = \left[\frac{875}{288}, \frac{875}{288}, \frac{875}{288}, 4, 5, 0\right],$$

$$u_8 = \left[\frac{1739}{576}, \frac{1739}{576}, \frac{1739}{576}, 4, 5, 0\right],$$

$$u_9 = \left[\frac{3467}{1152}, \frac{3467}{1152}, \frac{3467}{1152}, 4, 5, 0\right],$$

$$u_{10} = \left[\frac{6923}{2304}, \frac{6923}{2304}, \frac{6923}{2304}, 4, 5, 0\right],$$

$$u_{11} = \left[\frac{13835}{4608}, \frac{13835}{4608}, \frac{13835}{4608}, \frac{13835}{4608}, \frac{13835}{4608}, 0\right] \approx [3.002, 3.002, 3.002, 4, 5, 0].$$