4 Martingales in Discrete-Time

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Definition 4.1. A sequence $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is called a *filtration* if each $\mathcal{F}_n$ is a sub-σ-algebra of $\mathcal{F}$, and $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for $n = 0, 1, 2, \ldots$.

In other words, a filtration is an increasing sequence of sub-σ-algebras of $\mathcal{F}$. For notational convenience, we define

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$$

and note that $\mathcal{F}_\infty \subseteq \mathcal{F}$.

Definition 4.2. If $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is a filtration, and $X = \{X_n,\ n = 0, 1, \ldots\}$ is a discrete-time stochastic process, then X is said to be *adapted* to $\mathbb{F}$ if X_n is $\mathcal{F}_n$-measurable for each n.

Note that by definition every process $\{X_n,\ n = 0, 1, \ldots\}$ is adapted to its *natural filtration* $\{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ where $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.

The intuition behind the idea of a filtration is as follows. At time n, all of the information about $\omega \in \Omega$ that is known to us at time n is contained in $\mathcal{F}_n$. As n increases, our knowledge of ω also increases (or, if you prefer, does not decrease) and this is captured in the fact that the filtration consists of an increasing sequence of sub-σ-algebras.

Definition 4.3. A discrete-time stochastic process $X = \{X_n,\ n = 0, 1, \ldots\}$ is called a martingale (with respect to the filtration $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$) if for each $n = 0, 1, 2, \ldots$,

(a) X is adapted to $\mathbb{F}$; that is, X_n is $\mathcal{F}_n$-measurable,

(b) X_n is in L^1; that is, $\mathbb{E}|X_n| < \infty$, and

(c) $E(X_{n+1}|\mathcal{F}_n) = X_n$ a.s.

From this definition, we can immediately conclude the following useful facts.

Theorem 4.4. *Suppose that $X = \{X_n,\ n = 0, 1, \ldots\}$ is a martingale with respect to $\mathbb{F}$.*

(i) $\mathbb{E}(X_n) = \mathbb{E}(X_0)$ *for each* $n = 0, 1, 2, \ldots$.

(ii) $\mathbb{E}(X_n|\mathcal{F}_m) = X_m$ *for every non-negative integer* $m \le n$.

Proof. We begin first with the proof of (ii). Suppose that for a given n and m, we write $n = m + k$. Since $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is a filtration, repeated use of the tower property of conditional expectation gives

$$\begin{aligned}
\mathbb{E}(X_n|\mathcal{F}_m) &= \mathbb{E}(\mathbb{E}(X_n|\mathcal{F}_{n-1})|\mathcal{F}_m) = \mathbb{E}(X_{n-1}|\mathcal{F}_m) \\
&= \mathbb{E}(\mathbb{E}(X_{n-1}|\mathcal{F}_{n-2})|\mathcal{F}_m) = \mathbb{E}(X_{n-2}|\mathcal{F}_m) \\
&\ \vdots \\
&= \mathbb{E}(\mathbb{E}(X_{n-(k-2)}|\mathcal{F}_{n-(k-1)})|\mathcal{F}_m) = \mathbb{E}(X_{n-(k-1)}|\mathcal{F}_m) \\
&= \mathbb{E}(X_{m+1}|\mathcal{F}_m) = X_m.
\end{aligned}$$

As for (i), we see that taking $m = 0$ in (ii) gives $\mathbb{E}(X_n|\mathcal{F}_0) = X_0$. Thus taking expectations we find

$$\mathbb{E}(X_n) = \mathbb{E}(\mathbb{E}(X_n|\mathcal{F}_0)) = \mathbb{E}(X_0)$$

completing the proof. □

Remark. Some authors use part (ii) of the previous theorem in the definition of martingale in place of our (c). For those authors, part (c) then follows as an easy corollary of (ii) in the definition, just as for us (ii) followed as an easy corollary to our part (c).

4.1 Examples of martingales

Example 4.5. Suppose that $X_1, X_2, \ldots$ is a sequence of independent random variables with $\mathbb{E}|X_i| < \infty$ and $\mathbb{E}(X_i) = 0$ for each i. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and for $n \geq 1$ set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ so that $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is a filtration. Define the stochastic process S by setting $S_0 = 0$, and

$$S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1.$$

To show that $S = \{S_n,\ n = 0, 1, \ldots\}$ is a martingale with respect to $\mathbb{F}$ we need to verify the three parts of the definition. Clearly S_n is $\mathcal{F}_n$-measurable, and by assumption

$$\mathbb{E}|S_n| \leq \sum_{i=1}^{n} \mathbb{E}|X_i| < \infty.$$

Furthermore, since S_n is $\mathcal{F}_n$-measurable, and since X_{n+1} is independent of $\mathcal{F}_n$, we conclude

$$\mathbb{E}(S_{n+1}|\mathcal{F}_n) = \mathbb{E}(X_{n+1} + S_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1}|\mathcal{F}_n) + \mathbb{E}(S_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1}) + S_n = S_n.$$

Taken together these show that S is a martingale with respect to $\mathbb{F}$. Note that in this case $\mathbb{E}(S_n) = \mathbb{E}(S_0) = 0$ for each n.

Example 4.6. As a slight generalization of the previous example, suppose that Z is a random variable in L^1 which is independent of $X_1, X_2, \ldots$. Set $X_0 = Z$, and for $n = 0, 1, 2, \ldots$, take $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ and set

$$S_n = \sum_{i=0}^{n} X_i.$$

It follows as in the previous example that S is a martingale with respect to $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$, except this time $\mathbb{E}(S_n) = \mathbb{E}(S_0) = \mathbb{E}(Z)$ which need *not* equal 0.

Example 4.7. Suppose that S is as in Example 4.5, except assume further that $\mathbb{E}(X_i^2) < \infty$ for each i. We show that $Y = \{S_n^2,\ n = 0, 1, 2, \ldots\}$ is *not* a martingale with respect to $\mathbb{F}$. Although $Y_n = S_n^2$ is $\mathcal{F}_n$-measurable, and for each $n \geq 1$,

$$\mathbb{E}|Y_n| = \mathbb{E}(S_n^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2) < \infty,$$

we claim that $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) \neq Y_n$. Since X_{n+1} is independent of $\mathcal{F}_n$, and since S_n is $\mathcal{F}_n$-measurable, it follows that

$$\begin{aligned}
\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_{n+1}^2|\mathcal{F}_n) &= \mathbb{E}((S_{n+1} - S_n + S_n)^2|\mathcal{F}_n) \\
&= \mathbb{E}((S_{n+1} - S_n)^2|\mathcal{F}_n) - 2\mathbb{E}(S_n(S_{n+1} - S_n)|\mathcal{F}_n) + \mathbb{E}(S_n^2|\mathcal{F}_n) \\
&= \mathbb{E}(X_{n+1}^2|\mathcal{F}_n) - 2\mathbb{E}(X_{n+1} \cdot S_n|\mathcal{F}_n) + \mathbb{E}(S_n^2|\mathcal{F}_n) \\
&= \mathbb{E}(X_{n+1}^2) - 2S_n\mathbb{E}(X_{n+1}) + S_n^2 \\
&= 1 - 0 + S_n^2 = 1 + Y_n
\end{aligned}$$

Although Y_n is not a martingale, we see that $Z_n = S_n^2 - n$ is a martingale since

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_{n+1}^2|\mathcal{F}_n) - (n+1) = 1 + S_n^2 - (n+1) = S_n^2 - n = Z_n.$$

Example 4.8. Suppose that $X_1, X_2, \ldots$ is a sequence of independent, non-negative random variables with $\mathbb{E}(X_i) = 1$ for each i. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and for $n \geq 1$ set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ so that $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is a filtration. Define the stochastic process M by setting $M_0 = 1$, and

$$M_n = \prod_{i=1}^{n} X_i, \quad n \geq 1.$$

To show that $M = \{M_n,\ n = 0, 1, \ldots\}$ is a martingale with respect to $\mathbb{F}$ we need to verify the three parts of the definition. Clearly M_n is $\mathcal{F}_n$-measurable, and by assumption

$$\mathbb{E}|M_n| = \prod_{i=1}^{n} \mathbb{E}|X_i| = \prod_{i=1}^{n} \mathbb{E}(X_i) = 1 < \infty.$$

Furthermore, since M_n is $\mathcal{F}_n$-measurable, and since M_{n+1} is independent of $\mathcal{F}_n$, we conclude

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(X_{n+1} \cdot M_n|\mathcal{F}_n) = M_n \cdot \mathbb{E}(X_{n+1}|\mathcal{F}_n) = M_n \cdot \mathbb{E}(X_{n+1}) = M_n \cdot 1 = M_n.$$

Taken together these show that M is a martingale with respect to $\mathbb{F}$. Note that in this case $\mathbb{E}(M_n) = \mathbb{E}(M_0) = 1$ for each n.

Example 4.9. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is a filtration, and $Y : \Omega \to \mathbb{R}$ is a random variable in L^1. Define the stochastic process $X = \{X_n,\ n = 0, 1, \ldots\}$ by setting $X_n = \mathbb{E}(Y|\mathcal{F}_n)$ for each $n = 0, 1, 2, \ldots$. The definition of conditional expectation allows us to immediately conclude that X_n is $\mathcal{F}_n$-measurable, and from the conditional version of Jensen's inequality we have

$$\mathbb{E}|X_n| = \mathbb{E}|\mathbb{E}(Y|\mathcal{F}_n)| \leq \mathbb{E}(\mathbb{E}(|Y||\mathcal{F}_n)) = \mathbb{E}|Y| < \infty$$

so that $X_n \in L^1$ for each n. Furthermore, the tower property of conditional expectation gives

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = \mathbb{E}((\mathbb{E}(Y|\mathcal{F}_{n+1})|\mathcal{F}_n) = \mathbb{E}(Y|\mathcal{F}_n) = X_n$$

which shows that X is a martingale with respect to $\mathbb{F}$.

4.2 Stopping Times

Definition 4.10. A random variable $T : \Omega \to \mathbb{N} \cup \{+\infty\}$ is called a *stopping time* if $\{T \le n\} \in \mathcal{F}_n$ for every $n = 0, 1, 2, \ldots$.

Notice that the filtration $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is an integrable part of the definition. It is useful to think of a stopping time as the first time that a given random event happens with the convention that $T = +\infty$ if it never happens. If T is a stopping time, and $X = \{X_n,\ n = 0, 1, \ldots\}$ is a stochastic process, then define the random variable X_T as $X_{T(\omega)}(\omega)$.

Example 4.11. Suppose that S is a simple random walk on $\mathbb{Z}$, and let T be the first time that $S_n = 4$. That is,

$$T = \begin{cases} \min\{n \ge 0 : S_n = 4\}, & \text{if } S_n = 4 \text{ for some } n \in \mathbb{N}, \\ +\infty, & \text{otherwise,} \end{cases}$$

for in other words, $T(\omega) = \inf\{n \ge 0 : S_n(\omega) = 4\}$. (By writing inf, we stress that we follow the convention that the infimum of the empty set is $+\infty$.) In particular,

$$\{T \le n\} = \bigcup_{k=0}^{n} \{S_k = 4\} \in \mathcal{F}_n$$

where $\{S_k = 4\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$ for $k \le n$ because $\{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is a filtration. Hence, it follows that T is a stopping time.

Definition 4.12. A stopping time T is said to be *bounded* if there exists a constant $C < \infty$ such that $\mathbb{P}(T \le C) = 1$, and is said to be *finite a.s.* if $\mathbb{P}(T < \infty) = 1$.

Note that a bounded stopping time is necessarily finite a.s., although the converse need not be true. For example, consider repeatedly flipping a fair coin, and let T denote the first time a head is observed. Then T is finite a.s. but there is no C such that $\mathbb{P}(T \le C) = 1$. Indeed, $\mathbb{P}(T \le C) \le \mathbb{P}(T \le \lceil C \rceil) = 1 - 2^{-\lceil C \rceil} < 1$ for any constant $C < \infty$.

Definition 4.13. If T is a stopping time, then the *stopping time σ-algebra* $\mathcal{F}_T$ is defined as

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \le n\} \in \mathcal{F}_n \ \ \forall\, n\}.$$

Exercise 4.14. Show that $\mathcal{F}_T$ is, in fact, a σ-algebra.

Theorem 4.15. *A random variable $T : \Omega \to \mathbb{N} \cup \{+\infty\}$ is a stopping time if and only if $\{T = n\} \in \mathcal{F}_n$ for each $n = 0, 1, 2, \ldots$.*

Proof. Suppose that T is a stopping time so that $\{T \le n\} \in \mathcal{F}_n$ for each n. In particular, $\{T \le n-1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$ since $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, \ldots\}$ is a filtration so that

$$\{T = n\} = \{T \le n\} \cap \{T \le n-1\}^c \in \mathcal{F}_n.$$

On the other hand, if $\{T = n\} \in \mathcal{F}_n$ for each n, then $\{T = j\} \in \mathcal{F}_j \subseteq \mathcal{F}_n$ for each $j \le n$ since $\mathbb{F}$ is a filtration. Therefore,

$$\{T \le n\} = \bigcup_{j=0}^{n} \{T = j\} \in \mathcal{F}_n$$

so that T is a stopping time. □

Theorem 4.16. *If $X = \{X_n,\ n = 0, 1, \dots\}$ is a discrete-time stochastic process, and T is a stopping time, then the random variable X_T is $\mathcal{F}_T$-measurable.*

Proof. In order to prove that X_T is $\mathcal{F}_T$-measurable we must show that $\{X_T \in B\} \in \mathcal{F}_T$ for every Borel set $B \in \mathcal{B}(\mathbb{R})$. In other words, we must show that $\{X_T \in B\} \cap \{T \le n\} \in \mathcal{F}_n$ for every n. Since

$$\{X_T \in B\} \cap \{T \le n\} = \bigcup_{k=0}^{n} (\{X_T \in B\} \cap \{T = k\}) = \bigcup_{k=0}^{n} (\{X_k \in B\} \cap \{T = k\})$$

and since $\{X_k \in B\} \cap \{T = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$ for every $k \le n$, we conclude that X_T is $\mathcal{F}_T$-measurable as required. □

Exercise 4.17. If T is a stopping time, show that T is $\mathcal{F}_T$-measurable.

Recall from Theorem 4.4 that if $X = \{X_n,\ n = 0, 1, \dots\}$ is a martingale, then for *fixed* times $n = 0, 1, 2, \dots$ it follows that $\mathbb{E}(X_n) = \mathbb{E}(X_0)$. However, our goal is to determine when $\mathbb{E}(X_T) = \mathbb{E}(X_0)$ for *random* times T. Even when the random time T is a stopping time, this conclusion is not immediate as the following simple example shows.

Example 4.18. If S is a simple random walk on $\mathbb{Z}$, and $T = \inf\{n \ge 0 : S_n = 4\}$ as in Example 4.11, then clearly $S_T = 4$ so that $\mathbb{E}(S_T) = 4$. Since $\mathbb{E}(S_0) = 0$, we conclude that $\mathbb{E}(S_T) \neq \mathbb{E}(S_0)$.

Theorem 4.19. *If T is a bounded stopping time, and $X = \{X_n,\ n = 0, 1, \dots\}$ is a martingale, then $\mathbb{E}(X_T) = \mathbb{E}(X_0)$.*

Proof. Since T is a bounded stopping time, we may assume without loss of generality that T is bounded by N for some positive integer N. To begin, note that X_T can be written as

$$X_{T(\omega)}(\omega) = \sum_{n=0}^{\infty} X_n(\omega) \mathbb{1}_{\{T(\omega)=n\}}.$$

Since X is a martingale, we conclude that

$$\begin{aligned}
\mathbb{E}(X_T) &= \mathbb{E}\left(\sum_{n=0}^{\infty} X_n \mathbb{1}_{\{T=n\}}\right) = \mathbb{E}\left(\sum_{n=0}^{N} X_n \mathbb{1}_{\{T=n\}}\right) = \sum_{n=0}^{N} \mathbb{E}(X_n \mathbb{1}_{\{T=n\}}) \\
&= \sum_{n=0}^{N} \mathbb{E}(\mathbb{E}(X_N|\mathcal{F}_n)\mathbb{1}_{\{T=n\}}) = \sum_{n=0}^{N} \mathbb{E}(\mathbb{E}(X_N \mathbb{1}_{\{T=n\}}|\mathcal{F}_n)) = \sum_{n=0}^{N} \mathbb{E}(X_N \mathbb{1}_{\{T=n\}}) \\
&= \mathbb{E}\left(\sum_{n=0}^{N} X_N \mathbb{1}_{\{T=n\}}\right) = \mathbb{E}\left(X_N \sum_{n=0}^{N} \mathbb{1}_{\{T=n\}}\right) = \mathbb{E}(X_N) = \mathbb{E}(X_0)
\end{aligned}$$

where the last equality followed from Theorem 4.4. □