Stat 862: Stochastic Process
Winter 2006

Theorem: If $\Delta_n := \{a = t_0 < t_1 < \cdots t_{n-1} < t_n = b\}$ is a partition of $[a, b]$, then

$$\sum_{i=1}^{n} \left|B_{t_i} - B_{t_{i-1}}\right|^2 \to b - a \quad \text{in } L^2$$

as

$$||\Delta_n|| = \max_{1 \le i \le n} (t_i - t_{i-1}) \to 0.$$

That is, $Q_2(B; a, b)$, the quadratic variation of Brownian motion on the interval $[a, b]$, exists and equals $b - a$.

Proof: To begin, notice that

$$\sum_{i=1}^{n} (t_i - t_{i-1}) = b - a.$$

Let

$$Y_n = \sum_{i=1}^{n} \left|B_{t_i} - B_{t_{i-1}}\right|^2 - (b - a) = \sum_{i=1}^{n} \left[\left|B_{t_i} - B_{t_{i-1}}\right|^2 - (t_i - t_{i-1})\right] = \sum_{i=1}^{n} X_i$$

where

$$X_i = \left|B_{t_i} - B_{t_{i-1}}\right|^2 - (t_i - t_{i-1}),$$

and note that

$$Y_n^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j = \sum_{i=1}^{n} X_i^2 + 2 \sum_{i<j} X_i X_j.$$

The independence of the Brownian increments implies that $\mathbb{E}(X_i X_j) = 0$ for $i \neq j$; hence,

$$\mathbb{E}(Y_n^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2).$$

But

$$\begin{aligned}
\mathbb{E}(X_i^2) &= \mathbb{E}\left(B_{t_i} - B_{t_{i-1}}\right)^4 - 2(t_i - t_{i-1})\mathbb{E}\left(B_{t_i} - B_{t_{i-1}}\right)^2 + (t_i - t_{i-1})^2 \\
&= 3(t_i - t_{i-1})^2 - 2(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2 \\
&= 2(t_i - t_{i-1})^2
\end{aligned}$$

since the fourth moment of a normal random variable with mean 0 and variance $t_i - t_{i-1}$ is $3(t_i - t_{i-1})^2$. Therefore,

$$\mathbb{E}(Y_n^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2) = 2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 \le 2\, ||\Delta_n|| \sum_{i=1}^{n} (t_i - t_{i-1}) = 2(b - a)\, ||\Delta_n|| \to 0$$

as $||\Delta_n|| \to 0$ from which we conclude that $Y_n \to 0$ in L^2; that is, $Q_2(B; a, b) = b - a$. □