Statistics 862 Midterm – Winter 2006

This exam has 3 problems and is worth 30 points. **Instructor:** Michael Kozdron

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. This is a closed book exam; no aids are permitted.*

1. (*8 points*) Suppose that $M = \{M_n,\ n = 0, 1, 2, \ldots\}$ is a discrete time martingale with respect to the filtration $\mathbb{F} = \{\mathcal{F}_n,\ n = 0, 1, 2, \ldots\}$. Show that if M is previsible with respect to $\mathbb{F}$, then $M_n = M_0$ for each n.

2. (*8 points*) Let k and N be integers with $0 < k < N$, and suppose that S is a simple random walk on $\mathbb{Z}$ with $S_0 = k$. Consider the stopping time

$$T = \min\{j \geq 0 : S_j = 0 \quad \text{or} \quad S_j = N\}.$$

It can be shown that the optional sampling theorem can be applied to the martingale $Z_n = S_n^2 - n$. Use this fact to show that $E(T) = k(N - k)$.

Hint: You may want to use the fact that $P(S_T = N) = k/N$ which was shown in class.

3. (*14 points*)

(a) Suppose that $M = \{M_n,\ n = 0, 1, 2, \ldots\}$ is a discrete time martingale with $M_0 = 0$ and $\mathbb{E}(M_n^2) < \infty$ for each n. If $j < k < \ell < m$, show that

$$\mathbb{E}(\, M_j M_\ell (M_k - M_j)(M_m - M_\ell)\,) = 0.$$

(b) Let X_n, $n = 1, 2, \ldots$ be independent and identically distributed with $P(X_1 = 1) = P(X_1 = -1) = 1/2$. Set $S_0 = 0$, and for $n = 1, 2, \ldots$ define S_n to be

$$S_n = \sum_{i=1}^{n} X_i$$

so that $S = \{S_n,\ n = 0, 1, 2, \ldots\}$ is a simple random walk on $\mathbb{Z}$. Note that $\mathbb{E}(S_n^2) = n$ so that S satisfies **(a)**. For $j = 0, 1, 2, \ldots$, define

$$I_j = \sum_{n=1}^{j} S_{n-1}(S_n - S_{n-1}).$$

Show that $I = \{I_j,\ j = 0, 1, 2, \ldots\}$ is also a discrete time martingale with $I_0 = 0$.

(c) Let X, S, and I be as in part **(b)**. Show that $\operatorname{Var}(I_j) = \dfrac{j(j-1)}{2}$.

Hint: Recall that $\displaystyle\sum_{n=1}^{j} n = \frac{j(j+1)}{2}$.