Stat 862: Assignment #5

Problem #1: Suppose that $B = \{B_t,\ t \geq 0\}$ is a standard Brownian motion, and let $c > 0$ be a constant. Show that the process $Y = \{Y_t,\ t \geq 0\}$ defined by setting

$$Y_t = \frac{1}{c} B_{c^2 t}$$

is also a standard Brownian motion.

Problem #2: Let $B = \{B_t,\ t \geq 0\}$ be a standard Brownian motion with respect to the filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$.

(a) If $0 \leq q < r < s < t < \infty$, show that $\mathbb{E}(\,B_q B_s (B_t - B_s)(B_r - B_q)\,) = 0$. *Hint: Condition on $\mathcal{F}_s$, and use properties of both conditional expectations and Brownian motion.*

(b) Suppose that

$$I_n = \sum_{i=1}^{n} B_{\frac{i-1}{n}} \left(B_{\frac{i}{n}} - B_{\frac{i-1}{n}} \right).$$

Show that $\mathrm{Var}(I_n) = \dfrac{1}{2} - \dfrac{1}{2n}$.

Hint: Recall that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Problem #3: Let $B = \{B_t,\ t \geq 0\}$ be a standard Brownian motion with $B_0 = 0$. Fix $p > 0$, and for each $n = 1, 2, \ldots$, let $t_i = i/n$. Prove that

$$n^{p/2-1} \sum_{i=1}^{n} |B_{t_{i+1}} - B_{t_i}|^p$$

converges in probability to a constant μ_p as $n \to \infty$. *Hint: Use the scaling of Brownian motion, and the weak law of large numbers.*

Problem #4: Let $B = \{B_t,\ t \geq 0\}$ be a standard Brownian motion with $B_0 = 0$. Define the random variable X by setting

$$X(\omega) = \int_0^1 B_s(\omega)^2\, ds.$$

Compute $E(X)$ and $E(X^2)$.

Problem #5: Suppose that $B = \{B_t,\ t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, and for $a > 0$ let $T_a = \inf\{t \geq 0 : B_t = a\}$. We showed in class as a consequence of the reflection principle that

$$P(T_a \leq t) = 2 \int_a^\infty \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t} \right) dx.$$

Use the change of variables $x = a\, t^{1/2}\, s^{-1/2}$ to determine the density of T_a.