Stat 862: Assignment #2

Problem #1: Suppose X is a random variable and $\mathcal{F}$ is a σ-algebra such that

$$\mathbb{P}\{\, X \neq \mathbb{E}(X|\mathcal{F}) \,\} > 0.$$

(a) Show that for every $c \in \mathbb{R}$,

$$\mathbb{E}[\,|\,\mathbb{E}(X - c|\mathcal{F})\,|\,] \leq \mathbb{E}(\,|X - c|\,).$$

(b) Show that there exists some $c \in \mathbb{R}$,

$$\mathbb{E}[\,|\,\mathbb{E}(X - c|\mathcal{F})\,|\,] < \mathbb{E}(\,|X - c|\,).$$

(c) If X and Y are random variables, then we define $\mathbb{E}(X|Y)$ to be $\mathbb{E}(X|Y) = \mathbb{E}(X|\sigma(Y))$. Suppose that X and Y are such that $\mathbb{E}(X|Y) = Y$ a.s. and $\mathbb{E}(Y|X) = X$ a.s. Prove that $X = Y$ a.s. (*Hint*: Assume $X \neq Y$ and consider $X - c$ where c is as in **(b)** and $\mathcal{F} = \sigma(Y)$.)

Problem #2: Suppose that M_n is a discrete time martingale with respect to the filtration $\{\mathcal{F}_n\}$, and that $\mathbb{E}(M_n^2) < \infty$ for each n. Show that

$$\mathbb{E}[(M_n - M_0)^2] = \sum_{j=1}^{n} \mathbb{E}[(M_j - M_{j-1})^2].$$

Hint: You may first want to show that for every $j < k$,

$$\mathbb{E}[(M_j - M_{j-1})(M_k - M_{k-1})] = 0.$$

Problem #3: In this problem, we introduce the concept of a continuous time martingale. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A *filtration* $\{\mathcal{F}_t,\ t \geq 0\}$ is a collection of sub-σ-algebras of $\mathcal{F}$ with the property that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for every $0 \leq s \leq t$. A continuous time stochastic process $X = \{X_t,\ t \geq 0\}$ is said to be *adapted* to the filtration $\{\mathcal{F}_t,\ t \geq 0\}$ if for every $t \geq 0$, the random variable X_t is $\mathcal{F}_t$-measurable. We say that X is a *martingale* with respect to the filtration $\{\mathcal{F}_t,\ t \geq 0\}$ if

- X is adapted to $\{F_t,\ t \geq 0\}$,
- $\mathbb{E}(|X_t|) < \infty$ for every t, and
- $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for every $0 \leq s < t$.

(a) Suppose that $B = \{B_t,\ t \geq 0\}$ is a Brownian motion. Show that B is a martingale with respect to the natural Brownian filtration $\mathcal{F}_t = \sigma(B_s,\ s \leq t)$. As always, $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

(b) If B is a Brownian motion as in **(a)**, show that the process X defined by $X_t = B_t^2 - t$ is also a martingale with respect to the Brownian filtration.

(c) If B is a Brownian motion as in **(a)** and c is a constant, show that the process Y defined by

$$Y_t = \exp\left(cB_t - \frac{1}{2}c^2 t \right)$$

is also a martingale with respect to the Brownian filtration.