Stat 862: Solutions to Assignment #1

Problem #1: Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. Set $S_0 = 0$, and for $n \in \mathbb{N}$, let

$$S_n = \sum_{i=1}^{n} X_i$$

so that the stochastic process $\{S_n,\ n = 0, 1, 2, \ldots\}$ is a simple random walk on $\mathbb{Z}$. Compute the characteristic function of the random variable S_n.

Solution: Recall that the characteristic function of the random variable S_n is given by $\varphi_{S_n}(u) = \mathbb{E}(\exp(iuS_n))$. Thus, if $n = 0$, then $\varphi_{S_0}(u) = 1$, and for $n \geq 1$, we have

$$\begin{aligned}
\varphi_{S_n}(u) = \mathbb{E}\left(\exp\left(iu\sum_{i=1}^{n} X_i\right)\right) &= \prod_{i=1}^{n} \mathbb{E}\left(\exp(iuX_i)\right) \quad \text{since } X_i \text{ are independent} \\
&= [\,\mathbb{E}(\exp(iuX_1))\,]^n \quad \text{since } X_i \text{ are indentically distributed} \\
&= \left[\frac{1}{2}e^{iu} + \frac{1}{2}e^{-iu}\right]^n \\
&= \cos^n(u).
\end{aligned}$$

Problem #2: Suppose that $\Omega = [0, 1]$, $\mathcal{F}$ are the Borel sets of $[0, 1]$, and $\mathbb{P}$ is the uniform probability (i.e., Lebesgue measure) on $[0, 1]$, and assume that $\mathcal{F}$ is complete with respect to $\mathbb{P}$. For $t \in [0, 1]$, and $\omega \in [0, 1]$, define $X_t(\omega) = 0$ and $Y_t(\omega) = \mathbb{1}\{t = \omega\}$. Show that X and Y are versions of each other, but that they are not indistinguishable.

Solution: If $t \in [0, 1]$, then since $\mathbb{P}$ is the uniform probability, we see that $\mathbb{P}(X_t \neq Y_t) = \mathbb{P}(\{w : \omega = t\}) = 0$ so that X and Y are versions of each other. On the other hand, for every $\omega \in [0, 1]$, there exists a $t \in [0, 1]$ (namely $t = \omega$) such that the trajectory $Y(\omega)$ is not continuous at t. Therefore, $\mathbb{P}(X_t = Y_t\ \ \forall\ t) = 0$ so that X and Y are not indistinguishable.

Problem #3: Suppose that Y is a version of X, and that both X and Y have right-continuous sample paths. Show that X and Y are indistinguishable.

Solution: For each $t \geq 0$, let $Z_t = X_t - Y_t$ so that Z is a version of 0, and that Z has right-continuous sample paths. In order to show that Z is indistinguishable from 0, we must show that there exists a single null set N such if $\omega \notin N$, then $Z_t = 0\ \ \forall\ t$.

For each t, let $M_t = \{\omega : Z_t \neq 0\}$, and note that $\mathbb{P}(M_t) = 0$ since Z is a version of 0. Let

$$M = \bigcup_{t \in \mathbb{Q}} M_t$$

which has $\mathbb{P}(M) = 0$ by the countable subadditivity of $\mathbb{P}$. Finally, let $A = \{\omega : Z(\omega)$ is not right-continuous$\}$, and set $N = A \cup M$; hence $\mathbb{P}(N) = 0$.

(Recall that we write $Z(w)$ to denote the trajectory of $t \mapsto Z_t$ at ω.)

Note that $Z_t = 0$ for all $t \in \mathbb{Q}$ and $\omega \notin N$. On the other hand, suppose that $t \notin \mathbb{Q}$ and $\omega \notin N$, and let t_n be a sequence of rational numbers decreasing to t. Therefore, $Z_{t_n}(\omega) = 0$ for each $n = 1, 2, \ldots$, so by the right-continuity of the trajectory $Z(w)$ we conclude that

$$Z_t(\omega) = \lim_{n\to\infty} Z_{t_n}(\omega) = 0.$$

Thus, $\mathbb{P}(\{w : Z_t = 0 \ \forall \ t\}) = \mathbb{P}(N^c) = 1$ so that Z is indistinguishable from 0 as required.