Statistics 851 Midterm – October 11, 2013

This exam has 3 problems and is worth 40 points. **Instructor:** Michael Kozdron

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. This exam is closed-book, except that one* $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

1. (*16 points*) Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space. Suppose further that $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is a sequence of σ-algebras of Ω with the property that

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \cdots \subseteq \mathcal{F}.$$

Let

$$\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{F}_n.$$

(a) Prove that $\Omega \in \mathcal{A}$.

(b) Prove that $\mathcal{A}$ is closed under complements. That is, show that if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

(c) Prove that $\mathcal{A}$ has the finite intersection property. That is, show that if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

(d) Deduce from (a), (b), and (c) that $\mathcal{A}$ is an algebra of subsets of $\mathcal{F}$.

2. (*12 points*) Consider the uniform probability P on $([0,1], \mathcal{B}_1)$ where $\mathcal{B}_1$ denotes the Borel sets of $[0,1]$. In particular, $P([a,b]) = b - a$ for any $0 \le a < b \le 1$. Suppose that the events A_n and B_n, $n = 1, 2, \ldots$, are defined as

$$A_n = \left[0, \frac{1}{2} + \frac{(-1)^n}{2n}\right] \quad \text{and} \quad B_n = \left[\frac{1}{2} - \frac{(-1)^n}{2n}, 1\right].$$

(a) Compute $P(B_3|A_2)$ and $P(A_2|B_3)$.

(b) Compute $P\left(\bigcup_{n=1}^{\infty}(A_n \cap B_n)\right)$.

3. (*12 points*) Consider the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \begin{cases} 1 - \frac{1}{2}e^{-2x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

(a) Verify that F is actually a distribution function.

(b) Construct a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X : \Omega \to \mathbb{R}$ such that F_X, the distribution function of X, satisfies $F_X = F$.