Statistics 851: Probability
Fall 2005

Suppose that $(\Omega, \mathcal{A}, P)$ is a probability space with $\Omega = \{a, b, c, d, e\}$ and $\mathcal{A} = 2^\Omega$. Let X and Y be the real-valued random variables defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in \{a, b\}, \\ 0, & \text{if } \omega \notin \{a, b\}, \end{cases} \qquad Y(\omega) = \begin{cases} 2, & \text{if } \omega \in \{a, c\}, \\ 0, & \text{if } \omega \notin \{a, c\}. \end{cases}$$

(a) Give explicitly (by listing all the elements) the σ-algebras $\sigma(X)$ and $\sigma(Y)$ generated by X and Y, respectively.

Suppose that $B \in \mathcal{B}$ so that

$$X^{-1}(B) = \begin{cases} \Omega, & \text{if } 1 \in B, 0 \in B, \\ \{a, b\}, & \text{if } 1 \in B, 0 \notin B, \\ \{c, d, e\}, & \text{if } 1 \notin B, 0 \in B, \\ \emptyset, & \text{if } 1 \notin B, 0 \notin B. \end{cases}$$

Thus, the σ-algebra generated by X is $\sigma(X) = \{\emptyset, \Omega, \{a, b\}, \{c, d, e\}\}$. Similarly,

$$Y^{-1}(B) = \begin{cases} \Omega, & \text{if } 2 \in B, 0 \in B, \\ \{a, c\}, & \text{if } 2 \in B, 0 \notin B, \\ \{b, d, e\}, & \text{if } 2 \notin B, 0 \in B, \\ \emptyset, & \text{if } 2 \notin B, 0 \notin B. \end{cases}$$

so that $\sigma(Y) = \{\emptyset, \Omega, \{a, c\}, \{b, d, e\}\}$.

(b) Find the σ-algebra $\sigma(X, Y)$ generated (jointly) by X and Y.

By definition,

$$\begin{aligned} \sigma(X, Y) &= \sigma(\sigma(X), \sigma(Y)) = \sigma(\{a\}, \{b\}, \{c\}, \{d, e\}) \\ &= \{\emptyset, \Omega, \{a\}, \{b\}, \{c\}, \{d, e\}, \{a, b\}, \{a, c\}, \ \{a, d, e\}, \{b, c\}, \{b, d, e\}\{c, d, e\}, \{a, b, c\}, \\ &\qquad \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}. \end{aligned}$$

(c) If $Z = X + Y$, does $\sigma(Z) = \sigma(X, Y)$?

If $Z = X + Y$ then

$$Z(\omega) = \begin{cases} 3, & \text{if } \omega = a \\ 2, & \text{if } \omega = c \\ 1, & \text{if } \omega = b \\ 0, & \text{if } \omega \in \{d, e\} \end{cases}$$

so that

$$\sigma(Z) = \sigma(\{a\}, \{b\}, \{c\}, \{d, e\}) = \sigma(X, Y).$$

For real numbers $\alpha, \beta \geq 0$ with $\alpha+\beta \leq 1/2$, let P be the probability measure on $\mathcal{A}$ determined by the relations

$$P(\{a\}) = P(\{b\}) = \alpha, \quad P(\{c\}) = P(\{d\}) = \beta, \quad P(\{e\}) = 1 - 2(\alpha + \beta).$$

(d) Find all α, β for which $\sigma(X)$ and $\sigma(Y)$ are independent. Simplify!

Recall that $\sigma(X)$ and $\sigma(Y)$ are independent iff $P(A \cap B) = P(A)P(B)$ for every $A \in \sigma(X)$ and $B \in \sigma(Y)$. Notice that if either A or B are either $\emptyset$ or Ω, then $P(A \cap B) = P(A)P(B)$. Thus, in order to show that $\sigma(X)$ and $\sigma(Y)$ are independent, we need to simultaneously satisfy the four equalities.

$$P(\{a, b\} \cap \{a, c\}) = P(\{a, b\})P(\{a, c\}), \quad P(\{c, d, e\} \cap \{a, c\}) = P(\{c, d, e\})P(\{a, c\}),$$
$$P(\{a, b\} \cap \{b, d, e\}) = P(\{a, b\})P(\{b, d, e\}), \quad P(\{c, d, e\} \cap \{b, d, e\}) = P(\{c, d, e\})P(\{b, d, e\}).$$

By the definition of P, we have

$$P(\{a, b\}) = 2\alpha, \quad P(\{c, d, e\}) = 1 - 2\alpha, \quad P(\{a, c\}) = \alpha + \beta, \quad P(\{b, d, e\}) = 1 - \alpha - \beta$$

as well as

$$P(\{a, b\} \cap \{a, c\}) = P(\{a\}) = \alpha, \quad P(\{c, d, e\} \cap \{a, c\}) = P(\{c\}) = \beta,$$
$$P(\{a, b\} \cap \{b, d, e\}) = P(\{b\}) = \alpha, \quad P(\{c, d, e\} \cap \{b, d, e\}) = P(\{d, e\}) = 1 - 2\alpha - \beta.$$

Hence, our system of equations for α and β becomes

$$\begin{aligned} \alpha &= 2\alpha(\alpha + \beta) \\ \beta &= (1 - 2\alpha)(\alpha + \beta) \\ \alpha &= 2\alpha(1 - \alpha - \beta) \\ 1 - 2\alpha - \beta &= (1 - 2\alpha)(1 - \alpha - \beta) \end{aligned}$$

which has solution set

$$\left\{ (\alpha, \beta) : 0 \leq \alpha \leq \frac{1}{2}, \ 0 \leq \beta \leq \frac{1}{2}, \ \alpha + \beta = \frac{1}{2} \right\}.$$

(e) Find all α, β for which X and $Z = X + Y$ are independent.

In order for X and Z to be independent, we must have

$$P(X = i, Z = j) = P(X = i)P(Z = j)$$

for $i = 0, 1$, $j = 0, 1, 2, 3$. Notice, however, that

$$(X, Z) = \begin{cases} (1, 3), & \text{if } \omega \in \{a\}, \\ (1, 1), & \text{if } \omega \in \{b\}, \\ (0, 2), & \text{if } \omega \in \{c\}, \\ (0, 0), & \text{if } \omega \in \{d, e\}. \end{cases}$$

These imply that

$$P(\{a\}) = P(\{a, b\})P(\{a\}), \quad P(\{b\}) = P(\{a, b\})P(\{b\}),$$
$$P(\{c\}) = P(\{c, d, e\})P(\{c\}), \quad P(\{d, e\}) = P(\{c, d, e\}P(\{d, e\})$$

which are simultaneously satisfied by either $(\alpha = 1/2,\ \beta = 0)$ or $(\alpha = 0,\ \beta = 0)$.