Statistics 851: Probability
Fall 2005

Suppose that $(\Omega, \mathcal{A}, P)$ is a probability space with $\Omega = \{a, b, c, d, e\}$ and $\mathcal{A} = 2^\Omega$. Let X and Y be the real-valued random variables defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in \{a, b\}, \\ 0, & \text{if } \omega \notin \{a, b\}, \end{cases} \qquad Y(\omega) = \begin{cases} 2, & \text{if } \omega \in \{a, c\}, \\ 0, & \text{if } \omega \notin \{a, c\}. \end{cases}$$

(a) Give explicitly (by listing all the elements) the σ-algebras $\sigma(X)$ and $\sigma(Y)$ generated by X and Y, respectively.

(b) Find the σ-algebra $\sigma(X, Y)$ generated (jointly) by X and Y.

(c) If $Z = X + Y$, does $\sigma(Z) = \sigma(X, Y)$?

For real numbers α, $\beta \geq 0$ with $\alpha + \beta \leq 1/2$, let P be the probability measure on $\mathcal{A}$ determined by the relations

$$P(\{a\}) = P(\{b\}) = \alpha, \quad P(\{c\}) = P(\{d\}) = \beta, \quad P(\{e\}) = 1 - 2(\alpha + \beta).$$

(d) Find all α, β for which $\sigma(X)$ and $\sigma(Y)$ are independent. Simplify!

(e) Find all α, β for which X and $Z = X + Y$ are independent.