Statistics 452 Fall 2011 Midterm Exam – Solutions

1. Since $M_X(t) = \mathbb{E}(e^{tX})$ exists for $t \in (-h, h)$ for some $h > 0$ we know that

$$M_X'(0) = \mathbb{E}(X) \quad \text{and} \quad M_X''(0) = \mathbb{E}(X^2).$$

Moreover, it is always the case that $M_X(0) = 1$. From the chain rule, it follows that

$$\frac{d}{dt} \log M_X(t) = \frac{M_X'(t)}{M_X(t)}$$

and

$$\frac{d^2}{dt^2} \log M_X(t) = \frac{d}{dt}\left[\frac{M_X'(t)}{M_X(t)}\right] = \frac{M_X''(t)M_X(t) - M_X'(t)M_X'(t)}{M_X(t)^2}.$$

Therefore,

$$\frac{d}{dt} \log M_X(t)\bigg|_{t=0} = \frac{M_X'(0)}{M_X(0)} = M_X'(0) = \mathbb{E}(X)$$

and

$$\frac{d^2}{dt^2} \log M_X(t)\bigg|_{t=0} = \frac{M_X''(0)M_X(0) - M_X'(0)M_X'(0)}{M_X(0)^2} = M_X''(0) - [M_X'(0)]^2 = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$
$$= \text{Var}(X)$$

as required.

2. (a) If α is known and β is unknown, then

$$f(x|\beta) = \beta\alpha^\beta x^{-\beta-1} I(x > \alpha) = x^{-1} I(x > \alpha) \cdot \beta\alpha^\beta \cdot e^{-\beta \log x}.$$

Hence, if we take

$$h(x) = x^{-1} I(x > \alpha), \quad c(\beta) = \beta\alpha^\beta, \quad w(\beta) = \beta, \quad t(x) = -\log x,$$

then we see that

$$f(x|\beta) = h(x)c(\beta)e^{w(\beta)t(x)}$$

proving that $\{f(x|\beta) : \beta > 0\}$ does, in fact, form an exponential family.

2. (b) If α is unknown and β is known, then

$$f(x|\alpha) = \beta\alpha^\beta x^{-\beta-1} I(x > \alpha) = \beta x^{-\beta-1} \cdot \alpha^\beta \cdot I(x > \alpha).$$

Since the support of the density, namely $\{x : x > \alpha\}$, depends on the parameter α, we conclude that $\{f(x|\alpha) : \alpha > 0\}$ does not form an exponential family.

2. **(c)** In order to show that $\mathbb{E}(X^2)$ does not exist for $0 < \beta \le 2$, we will consider the integral

$$\int_{-\infty}^{\infty} x^2 f(x|\alpha,\beta)dx = \int_{\alpha}^{\infty} x^2 \cdot \beta\alpha^\beta x^{-\beta-1}dx = \beta\alpha^\beta \int_{\alpha}^{\infty} x^{1-\beta}dx. \qquad (*)$$

In order to evaluate the integral, there are three separate cases to treat. First, suppose that $\beta = 1$ so that

$$(*) = \alpha \int_{\alpha}^{\infty} dx = \alpha \lim_{N\to\infty} \int_{\alpha}^{N} dx = \alpha \lim_{N\to\infty} (N - \alpha) = \infty.$$

Now suppose that $\beta = 2$ so that

$$(*) = 2\alpha^2 \int_{\alpha}^{\infty} x^{-1}dx = 2\alpha^2 \lim_{N\to\infty} \int_{\alpha}^{N} x^{-1}dx = 2\alpha^2 \lim_{N\to\infty} (\log N - \log\alpha) = \infty.$$

Finally, suppose that $\beta \in (0,1) \cup (1,2)$ so that

$$\begin{aligned}(*) = \beta\alpha^\beta \int_{\alpha}^{\infty} x^{1-\beta}dx = \beta\alpha^\beta \lim_{N\to\infty} \int_{\alpha}^{N} x^{1-\beta}dx &= \beta\alpha^\beta \lim_{N\to\infty} \left.\frac{x^{2-\beta}}{2-\beta}\right|_{x=\alpha}^{x=N} \\ &= \frac{\beta\alpha^\beta}{2-\beta} \lim_{N\to\infty} \left(N^{2-\beta} - \alpha^{2-\beta}\right) \\ &= \infty\end{aligned}$$

since $\beta \in (0,1) \cup (1,2)$ implies $2 - \beta > 0$.

3. In order to show that $T(X_1, X_2) = X_1 + X_2$ is not sufficient for θ, we will show that for some choice of (x_1, x_2) and t, the ratio

$$\frac{P_\theta\{(X_1,X_2) = (x_1,x_2)\}}{P_\theta\{T(X_1,X_2) = t\}}$$

does depend on θ. Consider $t = 2$ so that

$$\begin{aligned}P_\theta\{T(X_1,X_2) = 2\} &= P_\theta\{X_1 + X_2 = 2\} \\ &= P_\theta\{(X_1,X_2) = (1,1)\} + P_\theta\{(X_1,X_2) = (0,2)\} + P_\theta\{(X_1,X_2) = (2,0)\} \\ &= \theta e^{-\theta} \cdot \theta e^{-\theta} + e^{-\theta} \cdot (1 - e^{-\theta} - \theta e^{-\theta}) + (1 - e^{-\theta} - \theta e^{-\theta}) \cdot e^{-\theta} \\ &= 2e^{-\theta} + e^{-2\theta}(\theta^2 - 2\theta - 2).\end{aligned}$$

Consider $(x_1, x_2) = (1, 1)$ so that

$$P_\theta\{(X_1,X_2) = (1,1)\} = \theta e^{-\theta} \cdot \theta e^{-\theta} = \theta^2 e^{-2\theta}.$$

Therefore,

$$\frac{P_\theta\{(X_1,X_2) = (1,1)\}}{P_\theta\{T(X_1,X_2) = 2\}} = \frac{\theta^2 e^{-2\theta}}{2e^{-\theta} + e^{-2\theta}(\theta^2 - 2\theta - 2)} = \frac{\theta^2}{2e^\theta + \theta^2 - 2\theta - 2}.$$

Since this ratio obviously depends on θ, we conclude that $T(X_1, X_2) = X_1 + X_2$ is not sufficient for θ.