Lecture #11: Continuity of Probability (continued)

We will now apply the continuity of probability theorem to prove that the function $F(x) = \mathbf{P}\{(-\infty, x]\}$, $x \in \mathbb{R}$, defined last lecture is actually a distribution function.

Theorem 11.1. *Consider the real numbers $\mathbb{R}$ with the Borel σ-algebra $\mathcal{B}$, and let $\mathbf{P}$ be a probability on $(\mathbb{R}, \mathcal{B})$. The function $F : \mathbb{R} \to [0, 1]$ defined by $F(x) = \mathbf{P}\{(-\infty, x]\}$ for $x \in \mathbb{R}$ is a distribution function; that is,*

(i) $\lim_{x\to-\infty} F(x) = 0$ *and* $\lim_{x\to\infty} F(x) = 1$,

(ii) F is right-continuous, and

(iii) F is increasing.

Proof. In order to prove that $F(x) = \mathbf{P}\{(-\infty, x]\}$ is a distribution function we need to verify that the three conditions in definition are met. We will begin by showing (ii). Thus, to show that F is right-continuous, we must show that if x_n is a sequence of real numbers converging to x from the right, i.e., $x_n \downarrow x$ or $x_n \to x+$, then $F(x_n)$ converges to $F(x)$, i.e.,

$$\lim_{x_n\to x+} F(x_n) = F(x).$$

However, this follows immediately from the continuity of probability theorem (actually it follows directly from Exercise 10.3 which follows directly from Theorem 10.2) by noting that if $x_n \to x+$, then

$$(-\infty, x_1] \supseteq (-\infty, x_2] \supseteq \cdots \supseteq (-\infty, x_j] \supseteq (-\infty, x_{j+1}] \supseteq \cdots \supseteq (-\infty, x]$$

and

$$\bigcap_{n=1}^{\infty} (-\infty, x_n] = (-\infty, x]$$

so that

$$\lim_{x_n\to x+} F(x_n) = \lim_{n\to\infty} \mathbf{P}\{(-\infty, x_n]\} = \mathbf{P}\left\{\bigcap_{n=1}^{\infty} (-\infty, x_n]\right\} = \mathbf{P}\{(-\infty, x]\} = F(x).$$

It now follows from (ii) and the fact that $\mathbf{P}$ is a probability on $(\mathbb{R}, \mathcal{B})$ that

$$\lim_{x\to-\infty} F(x) = \mathbf{P}\{(-\infty, -\infty)\} = \mathbf{P}\{\emptyset\} = 0$$

and

$$\lim_{x\to\infty} F(x) = \mathbf{P}\{(-\infty, \infty)\} = \mathbf{P}\{\mathbb{R}\} = 1.$$

This establishes (i). To show that F is increasing, observe that if $x \le y$, then $(-\infty, x] \subseteq (-\infty, y]$. Since $\mathbf{P}$ is a probability, this implies that $F(x) = \mathbf{P}\{(-\infty, x]\} \le \mathbf{P}\{(-\infty, y]\} = F(y)$. This establishes (iii) and taken together the proof is complete. □

A first look at random variables

Consider a chance experiment. We have defined a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ consisting of a sample space Ω of outcomes, a σ-algebra $\mathcal{F}$ of events, and an assignment $\mathbf{P}$ of probabilities to events as a model for the experiment. It is often the case that one is not interested in a particular outcome per se, but rather in a function of the outcome. This is readily apparent if we consider a bet on a game of chance at a casino. For instance, suppose that a gambler pays \$3 to roll a fair die and then wins \$$j$ where j is the side that appears, $j = 1, \ldots, 6$. Hence, the gambler's net income is either $-\$2$, $-\$1$, \$0, \$1, \$2, or \$3 depending on whether a 1, 2, 3, 4, 5, or 6 appears. If we let $\Omega = \{1, 2, 3, 4, 5, 6\}$ denote the sample space for this experiment, and we let X denote the gambler's net income, then it is clear that X is the real-valued function on Ω given by

$$X(1) = -2, \quad X(2) = -1, \quad X(3) = 0, \quad X(4) = 1, \quad X(5) = 2, \quad X(6) = 3.$$

More succinctly, we might write $X : \Omega \to \mathbb{R}$ defined by $X(\omega) = \omega - 3$. The function X is an example of a *random variable*. Observe that

$$\mathbf{P}\{X = -2\} = \mathbf{P}\{\omega \in \Omega : X(\omega) = -2\} = \mathbf{P}\{1\} = \frac{1}{6},$$

and, similarly,

$$\mathbf{P}\{X = -1\} = \mathbf{P}\{X = 0\} = \mathbf{P}\{X = 1\} = \mathbf{P}\{X = 2\} = \mathbf{P}\{X = 3\} = \frac{1}{6}.$$

Thus, to understand the likelihood of having a certain net winning, it is enough to know the probabilities of the outcomes associated with that net winning.

This leads to the general notion of a random variable as a real-valued function on Ω. As we will see shortly, the sort of trouble that we had with constructing the uniform probability on the uncountable space $([0, 1], \mathcal{B}_1)$ is the same sort of trouble that will prevent *any* real-valued function on Ω from being a random variable. It will turn out that only a special type of function, known as a measurable function, will be a random variable. Fortunately, every *reasonable* function (including those that one is likely to encounter when applying probability theory to everyday chance experiments such as casino games) will be measurable. For a function not to be measurable, it will need to be *really weird*.

The definition of random variable

Suppose that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. As in the example above, we want to compute probabilities associated with certain values of the random variable; that is, we want to compute $\mathbf{P}\{X \in B\}$ for any Borel set B.

Hence, if we want to be able to compute $\mathbf{P}\{X \in B\} = \mathbf{P}\{\omega : X(\omega) \in B\} = \mathbf{P}\{X^{-1}(B)\}$ for every Borel set B, then it must be the case that $X^{-1}(B)$ is an event (which is to say that $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}$).

Definition. A real-valued function $X : \Omega \to \mathbb{R}$ is said to be a *random variable* if $X^{-1}(B) \in \mathcal{F}$ for every Borel set $B \in \mathcal{B}$.

Note that when we say *let X be a random variable*, we really mean *let X be a function from the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ to the real numbers endowed with the Borel σ-algebra $(\mathbb{R}, \mathcal{B})$ such that $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}$.* Hence, when we define a random variable, we should really also state the underlying probability space as the domain space of X. Since every random variable we will consider is real-valued, our codomain (or target) space will always be $\mathbb{R}$ endowed with the Borel σ-algebra $\mathcal{B}$. If we want to stress the domain space and codomain space, we will be explicit and write $X : (\Omega, \mathcal{F}, \mathbf{P}) \to (\mathbb{R}, \mathcal{B})$.

Example 11.2. Perhaps the simplest example of a random variable is the indicator function of an event. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and suppose that $A \in \mathcal{F}$ is an event. Let $X : \Omega \to \mathbb{R}$ be given by

$$X(\omega) = 1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

For $B \in \mathcal{B}$, we find

$$(1_A)^{-1}(B) = \begin{cases} \emptyset, & \text{if } 0 \notin B,\ 1 \notin B, \\ A, & \text{if } 0 \notin B,\ 1 \in B, \\ A^c, & \text{if } 0 \in B,\ 1 \notin B, \\ \Omega, & \text{if } 0 \in B,\ 1 \in B. \end{cases}$$

Thus, since $\emptyset$, A, A^c, and Ω belong to $\mathcal{F}$, we see that for any $B \in \mathcal{B}$ we necessarily have $X^{-1}(B) = (1_A)^{-1}(B) \in \mathcal{F}$ proving that X is a random variable.