Statistics 451 (Fall 2013) September 25, 2013
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Lecture #10: Continuity of Probability

Recall that last class we proved the following theorem.

Theorem 10.1. *Consider the real numbers* $\mathbb{R}$ *with the Borel* σ*-algebra* $\mathcal{B}$*, and let* $\mathbf{P}$ *be a probability on* $(\mathbb{R}, \mathcal{B})$*. The function* $F : \mathbb{R} \to [0,1]$ *defined by* $F(x) = \mathbf{P}\{(-\infty, x]\}$*,* $x \in \mathbb{R}$*, characterizes* $\mathbf{P}$*.*

The function F in the statement of the theorem is an example of a *distribution function* and will be of fundamental importance when we study random variables later on in the course.

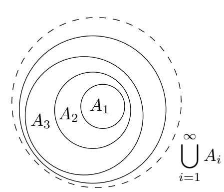
Definition. A function $F : \mathbb{R} \to [0,1]$ is called a *distribution function* if

(i) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$,

(ii) F is right-continuous, and

(iii) F is increasing.

Before we continue our discussion of distribution functions, we will need a result known as the *continuity of probability* theorem. If A_j, $j = 1, 2, \ldots$, is a sequence of events, then we say that A_j *increases* to A and write $\{A_j\} \uparrow A$ if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and $\bigcup_{j=1}^{\infty} A_j = A$. If B_j, $j = 1, 2, \ldots$, is a sequence of events, then we say that B_j *decreases* to B and write $\{B_j\} \downarrow B$ if $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ and $\bigcap_{j=1}^{\infty} B_j = B$. Note that B_j decreases to B if and only if B_j^c increases to B^c.



Theorem 10.2. *If* A_j*,* $j = 1, 2, \ldots$*, is a sequence of events increasing to* A*, then*

$$\lim_{n\to\infty} \mathbf{P}\{A_n\} = \mathbf{P}\{A\}.$$

Proof. Since $A_j \subseteq A_{j+1}$ we see that $A_j \cap A_{j+1} = A_j$. Therefore, we consider the event $C_{j+1} = A_{j+1} \cap A_j^c$, namely that part of A_{j+1} not in A_j. For notational convenience, take $A_0 = \emptyset$ so that $C_1 = A_1$. Notice that $C_1, C_2, \ldots$ are disjoint with

$$\bigcup_{j=1}^{n} C_j = A_n \quad \text{and} \quad \bigcup_{j=1}^{\infty} C_j = \bigcup_{j=1}^{\infty} A_j = A.$$

Therefore, using the fact that $\mathbf{P}$ is countably additive, we conclude

$$\mathbf{P}\{A\} = \mathbf{P}\left\{\bigcup_{j=1}^{\infty} C_j\right\} = \sum_{j=1}^{\infty} \mathbf{P}\{C_j\} = \lim_{n\to\infty} \sum_{j=1}^{n} \mathbf{P}\{C_j\} = \lim_{n\to\infty} \mathbf{P}\left\{\bigcup_{j=1}^{n} C_j\right\} = \lim_{n\to\infty} \mathbf{P}\{A_n\}$$

as required. □

Exercise 10.3. Show that if B_j, $j = 1, 2, \ldots$, is a sequence of events decreasing to B, then

$$\lim_{n\to\infty} \mathbf{P}\{B_n\} = \mathbf{P}\{B\}.$$

One important application of the continuity of probability theorem is the following. This result is usually known as the Borel-Cantelli Lemma. (Actually, it is usually given as the first part of the Borel-Cantelli Lemma.)

Theorem 10.4. *Suppose that A_j, $j = 1, 2, \ldots$, is a sequence of events. If*

$$\sum_{j=1}^{\infty} \mathbf{P}\{A_j\} < \infty, \tag{10.3}$$

then

$$\mathbf{P}\left\{\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right\} = 0.$$

Proof. For $n = 1, 2, \ldots$, if we define

$$B_n = \bigcup_{m=n}^{\infty} A_m,$$

then B_n, $n = 1, 2, \ldots$, is a sequence of decreasing events and so by Exercise 10.3 we find

$$\mathbf{P}\left\{\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right\} = \mathbf{P}\left\{\bigcap_{n=1}^{\infty} B_n\right\} = \lim_{n\to\infty} \mathbf{P}\{B_n\}.$$

Using countable subadditivity, we see that

$$\mathbf{P}\{B_n\} = \mathbf{P}\left\{\bigcup_{m=n}^{\infty} A_m\right\} \le \sum_{m=n}^{\infty} \mathbf{P}\{A_m\}.$$

The hypothesis (10.3) implies that

$$\lim_{n\to\infty} \sum_{m=n}^{\infty} \mathbf{P}\{A_m\} = 0$$

and so the proof is complete. □

Remark. The event

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

occurs so frequently in probability that it is often known as the event "A_n infinitely often" or as the limit supremum of the A_n. That is,

$$\limsup_{n\to\infty} A_n = \{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Exercise 10.5. One of our requirements when we defined a probability was that it be countably additive. It would have been equivalent to replace that condition with the requirement that it be *continuous* in the sense of the previous theorem. That is, suppose that Ω is a sample space and $\mathcal{F}$ is a σ-algebra of subsets of Ω. Show that if $\mathbf{P} : \mathcal{F} \to [0,1]$ is a set function with $\mathbf{P}\{\Omega\} = 1$ and such that

$$\mathbf{P}\left\{\bigcup_{j=1}^{n} A_j\right\} = \sum_{j=1}^{n} \mathbf{P}\{A_j\}$$

whenever $A_1, \ldots, A_n \in \mathcal{F}$ are disjoint, then the following two statements are equivalent.

(i) If $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, then

$$\mathbf{P}\left\{\bigcup_{j=1}^{\infty} A_j\right\} = \sum_{j=1}^{\infty} \mathbf{P}\{A_j\}.$$

(ii) If $A_1, A_2, \ldots \in \mathcal{F}$ and $\{A_j\} \uparrow A$, then

$$\lim_{n\to\infty} \mathbf{P}\{A_n\} = \mathbf{P}\{A\}.$$

(Note that (i) implies (ii) by the continuity of measure theorem. The problem is really to show that (ii) implies (i).)