Lecture #9: Construction of a Probability (Part II)

Recall that we have been trying to construct a uniform probability on $[0, 1]$. As we saw in Lecture #4, it is not possible to construct a uniform probability for every subset of $[0, 1]$. This meant that $2^{[0,1]}$ was too big a σ-algebra to use. We then discussed the Borel sets and indicated that the Borel σ-algebra is the "right" one to use. Today we will formalize that discussion.

Also recall from Lecture #6 that we had the following corollary to the monotone class theorem.

Corollary 9.1. *Let $\mathcal{F}$ be a σ-algebra, and let $\mathbf{P}$, $\mathbf{Q}$ be two probabilities on $(\Omega, \mathcal{F})$. Suppose that $\mathbf{P}$, $\mathbf{Q}$ agree on a class $\mathcal{C} \subseteq \mathcal{F}$ which is closed under finite intersections. If $\sigma(\mathcal{C}) = \mathcal{F}$, then $\mathbf{P} = \mathbf{Q}$.*

In particular, note that if $\mathcal{F}_0$ is an algebra, then $\mathcal{F}_0$ is necessarily closed under finite intersections.

Theorem 9.2. *Consider the real numbers $\mathbb{R}$ with the Borel σ-algebra $\mathcal{B}$, and let $\mathbf{P}$ be a probability on $(\mathbb{R}, \mathcal{B})$. The function $F : \mathbb{R} \to [0, 1]$ defined by $F(x) = \mathbf{P}\{(-\infty, x]\}$, $x \in \mathbb{R}$, characterizes $\mathbf{P}$.*

Proof. Let $\mathcal{B}_0$ denote the set of finite disjoint unions of intervals of the form $(x, y]$ for $-\infty \le x \le y \le \infty$ following the convention that $(x, \infty] = (x, \infty)$. It is easily checked that $\mathcal{B}_0$ is an algebra.

As we will now show, $\sigma(\mathcal{B}_0) = \mathcal{B}$. To begin, observe that

$$(a, b) = \bigcup_{n=N}^{\infty} \left(a, b - \frac{1}{n}\right]$$

for some sufficiently large N. This implies that $\mathcal{B} \subseteq \sigma(\mathcal{B}_0)$. Moreover,

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$$

so that $\mathcal{B}_0 \subseteq \mathcal{B}$ and therefore $\sigma(\mathcal{B}_0) \subseteq \mathcal{B}$. Taken together, we have $\sigma(\mathcal{B}_0) = \mathcal{B}$.

We now observe that

$$\mathbf{P}\{(x, y]\} = \mathbf{P}\{(-\infty, y] \setminus (-\infty, x]\} = \mathbf{P}\{(-\infty, y]\} - \mathbf{P}\{(-\infty, x]\} = F(y) - F(x)$$

and so if $A \in \mathcal{B}_0$ is of the form

$$A = \bigcup_{i=1}^{n} (x_i, y_i]$$

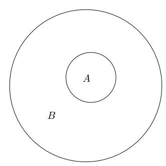
with $y_i < x_{i+1}$, then

$$\mathbf{P}\{A\} = \sum_{i=1}^{n}[F(y_i) - F(x_i)].$$

Suppose that $\mathbf{Q}$ is another probability on $(\Omega, \mathcal{F})$ satisfying $F(x) = \mathbf{Q}\{(-\infty, x]\}$. Repeating the argument just given shows that $\mathbf{Q}\{A\} = \mathbf{P}\{A\}$ for every $A \in \mathcal{B}_0$. Since $\mathcal{B}_0$ is an algebra and $\mathbf{Q} = \mathbf{P}$ on $\mathcal{B}_0$, we conclude from the corollary to the monotone class theorem that $\mathbf{Q} = \mathbf{P}$ on $\mathcal{B}$. □

Definition. Let $\mathbf{P}$ be a probability on $\mathcal{F}$. A *null set* (or a *negligible set*) for $\mathbf{P}$ is a subset $A \subseteq \Omega$ such that there exists a $B \in \mathcal{F}$ with $A \subseteq B$ and $\mathbf{P}\{B\} = 0$.

Note. Suppose that $B \in \mathcal{F}$ with $\mathbf{P}\{B\} = 0$. Let $A \subseteq B$ as shown below.



If $A \in \mathcal{F}$, then we can conclude that $\mathbf{P}\{A\} = 0$. However, if $A \notin \mathcal{F}$, then $\mathbf{P}\{A\}$ does not make sense.

In either case, A is a null set. Thus, it is natural to *define* $\mathbf{P}\{A\} = 0$ for all null sets.

Theorem 9.3. *Suppose that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space so that $\mathbf{P}$ is a probability on the σ-algebra $\mathcal{F}$. Let $\mathcal{N}$ denote the set of all null sets for $\mathbf{P}$. If*

$$\mathcal{F}' = \mathcal{F} \cup \mathcal{N} = \{A \cup N : A \in \mathcal{F}, N \in \mathcal{N}\},$$

then $\mathcal{F}'$ is a σ-algebra, called the $\mathbf{P}$-completion *of $\mathcal{F}$, and is the smallest σ-algebra containing $\mathcal{F}$ and $\mathcal{N}$. Furthermore, $\mathbf{P}$ extends uniquely to a probability on $\mathcal{F}'$ (denoted by $\mathbf{P}'$) by setting*

$$\mathbf{P}'\{A \cup N\} = \mathbf{P}\{A\}$$

for $A \in \mathcal{F}$, $N \in \mathcal{N}$.

Proof. To show that $\mathcal{F}'$ is a σ-algebra, we need to verify that the three conditions in the definition of σ-algebra are met. To that end, we will first show that $\Omega \in \mathcal{F}'$. Since $\emptyset \in \mathcal{F}$ has $\mathbf{P}\{\emptyset\} = 0$ and $\emptyset \subseteq \emptyset$, we conclude that $\emptyset \in \mathcal{N}$ is a null set. If we now write $\Omega = \Omega \cup \emptyset$ then we have expressed Ω as a union of an event in $\mathcal{F}$ (namely Ω) and a null set (namely $\emptyset$). This shows $\Omega \in \mathcal{F}'$. We will now show that $\mathcal{F}'$ is closed under complements. That is, suppose $E \in \mathcal{F}'$ so that $E = A \cup N$ for some event $A \in \mathcal{F}$ and some null set $N \in \mathcal{N}$. Since N is a null set, we know there exists some event $B \in \mathcal{F}$ with $\mathbf{P}\{B\} = 0$. We now observe that $N^c = B^c \cup (B \setminus N)$ and so

$$E^c = A^c \cap N^c = (A^c \cap B^c) \cup (N^c \cap (B \setminus N))$$

Since $\mathcal{F}$ is a σ-algebra and A, $B \in F$, we know that $A^c \cap B^c \in \mathcal{F}$. Moreover, we know that $(N^c \cap (B \setminus N))$ is a null set since $(N^c \cap (B \setminus N)) \subseteq B \setminus N \subseteq B$. This shows that $E^c \in \mathcal{F}'$ since E^c can be expressed as the union of an event in $\mathcal{F}$ and a null set. Finally, suppose that $E_1, E_2, \ldots, \in \mathcal{F}'$ are disjoint so that $E_j = A_j \cup N_j$ where $A_1, A_2, \ldots,$ is a sequence of disjoint events and $N_1, N_2, \ldots$ are null sets. Since N_j is a null set, there exist events $B_j \in \mathcal{F}$, $j = 1, 2, \ldots$, with $N_j \subseteq B_j$ and $\mathbf{P}\{B_j\} = 0$. Since

$$\bigcup_{j=1}^{\infty} N_j \subseteq \bigcup_{j=1}^{\infty} B_j$$

and countable subadditivity of $\mathbf{P}$ implies

$$\mathbf{P}\left\{\bigcup_{j=1}^{\infty} B_j\right\} \le \sum_{j=1}^{\infty} \mathbf{P}\{B_j\} = 0,$$

we see that $\cup_{j=1}^{\infty} N_j$ is a null set. Therefore,

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (A_j \cup N_j) = \left(\bigcup_{j=1}^{\infty} A_j\right) \cup \left(\bigcup_{j=1}^{\infty} N_j\right) \in \mathcal{F}' \tag{9.3}$$

since $\cup_{j=1}^{\infty} A_j \in \mathcal{F}$ and $\cup_{j=1}^{\infty} N_j$ is a null set. We also note that the fact that $\mathcal{F}' = \mathcal{F} \cup \mathcal{N}$ is a σ-algebra implies that $\mathcal{F}'$ must be the smallest σ-algebra containing $\mathcal{F}$ and $\mathcal{N}$. Finally, to show that $\mathbf{P}'$ is a probability on $(\Omega, \mathcal{F}')$, we begin by noting that

$$\mathbf{P}'\{\Omega\} = \mathbf{P}'\{\Omega \cup \emptyset\} = \mathbf{P}\{\Omega\} = 1.$$

Moreover, using (9.3), we find that if $E_1, E_2, \ldots \in \mathcal{F}'$ are disjoint, then

$$\begin{aligned}\mathbf{P}'\left\{\bigcup_{j=1}^{\infty} E_j\right\} = \mathbf{P}'\left\{\left(\bigcup_{j=1}^{\infty} A_j\right) \cup \left(\bigcup_{j=1}^{\infty} N_j\right)\right\} = \mathbf{P}\left\{\bigcup_{j=1}^{\infty} A_j\right\} &= \sum_{j=1}^{\infty} \mathbf{P}\{A_j\} \\ &= \sum_{j=1}^{\infty} \mathbf{P}'\{A_j \cup N_j\} \\ &= \sum_{j=1}^{\infty} \mathbf{P}'\{E_j\}\end{aligned}$$

and the proof is complete. □

Application: Construction of the Uniform Probability on $([0,1], \mathcal{B}_1)$

Recall from Lecture #5 that $\mathcal{B}_1$ denotes the Borel sets of $[0,1]$; that is, the σ-algebra generated by the open subsets of $[0,1]$. Also recall that one condition we want the uniform probability to satisfy is that the probability of an interval $(a,b] \subseteq [0,1]$ is equal to its length;

that is, if $0 \leq a < b \leq 1$, we want $\mathbf{P}\{(a,b]\} = b - a$. This corresponds to the function $F : \mathbb{R} \to [0,1]$ given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1, \end{cases}.$$

By Theorem 9.2 we know F uniquely classifies the uniform probability on $([0,1], \mathcal{B}_1)$.

We remark in closing that if $\mathbf{P}$ denotes the uniform probability on $([0,1], \mathcal{B}_1)$, then one can consider the σ-algebra

$$\mathcal{M}_1 = \mathcal{B}_1 \cup \mathcal{N}$$

where $\mathcal{N}$ denotes the $\mathbf{P}$-null sets and $\mathcal{M}_1$ is known as the σ-algebra of Lebesgue measurable sets of $[0,1]$. Thus, we have the chain of containments

$$\mathcal{B}_1 \subseteq \mathcal{M}_1 \subsetneq 2^{[0,1]}.$$

The question of whether or not $\mathcal{B}_1$ and $\mathcal{M}_1$ are equal remained open for many years. It was shown by Lusin in 1927 that $\mathcal{B}_1 \subsetneq \mathcal{M}_1$.

Remark. In courses on measure theory (such as Math 810), the uniform probability on $([0,1], \mathcal{B}_1)$ is also known as *Lebesgue measure*. In those courses, however, the usual approach is to construct the Lebesgue measure on $\mathcal{M}_1$ directly and then restrict it to $\mathcal{B}_1$.