Lecture #7: Proof of the Monotone Class Theorem

Our goal for today is to prove the monotone class theorem. We will then deduce an extremely important corollary which we will ultimately use to prove that one can construct the uniform probability on $[0, 1]$ with the Borel σ-algebra.

Theorem 7.1 (Monotone Class Theorem). *Let Ω be a sample space, and let $\mathcal{C}$ be a class of subsets of Ω. Suppose that $\mathcal{C}$ is closed under finite intersections and that $\mathcal{C}$ contains Ω (that is, $\Omega \in \mathcal{C}$). If $\mathcal{D}$ is the smallest class containing $\mathcal{C}$ which is closed under increasing limits and finite differences, then*

$$\mathcal{D} = \sigma(\mathcal{C}).$$

Proof. We begin by noting that the intersection of classes of sets closed under increasing limits and finite differences is again a class of that type. Hence, if we take the intersection of all such classes, then there will be a smallest class containing $\mathcal{C}$ which is closed under increasing limits and by finite differences. Denote this class by $\mathcal{D}$. Also note that a σ-algebra is necessarily closed under increasing limits and by finite differences. Thus, we conclude that $\mathcal{D} \subseteq \sigma(\mathcal{C})$. To complete the proof we will show the reverse containment, namely $\sigma(\mathcal{C}) \subseteq \mathcal{D}$.

For every set $B \subseteq \Omega$, let

$$\mathcal{D}_B = \{A \subseteq \Omega : A \in \mathcal{D} \text{ and } A \cap B \in \mathcal{D}\}.$$

Since $\mathcal{D}$ is closed under increasing limits and finite differences, a calculation shows that $\mathcal{D}_B$ must also closed under increasing limits and finite differences.

Since $\mathcal{C}$ is closed under finite intersections, $\mathcal{C} \subseteq \mathcal{D}_B$ for every $B \in \mathcal{C}$. That is, suppose that $B \in \mathcal{C}$ is fixed and let $C \in \mathcal{C}$ be arbitrary. Since $\mathcal{C}$ is closed under finite intersections, we must have $B \cap C \in \mathcal{C}$. Since $\mathcal{C} \subseteq \mathcal{D}$, we conclude that $B \cap C \in \mathcal{D}$ verifying that $C \in \mathcal{D}_B$ for every $B \in \mathcal{C}$. Note that by definition we have $\mathcal{D}_B \subseteq \mathcal{D}$ for every $B \in \mathcal{C}$ and so we have shown

$$\mathcal{C} \subseteq \mathcal{D}_B \subseteq \mathcal{D}$$

for every $B \in \mathcal{C}$. Since $\mathcal{D}_B$ is closed under increasing limits and finite differences, we conclude that $\mathcal{D}$, the smallest class containing $\mathcal{C}$ closed under increasing limits and finite differences, must be contained in $\mathcal{D}_B$ for every $B \in \mathcal{C}$. That is, $\mathcal{D} \subseteq \mathcal{D}_B$ for every $B \in \mathcal{C}$. Taken together, we are forced to conclude that $\mathcal{D} = \mathcal{D}_B$ for every $B \in \mathcal{C}$.

Now suppose that $A \in \mathcal{D}$ is arbitrary. We will show that $\mathcal{C} \subseteq \mathcal{D}_A$. If $B \in \mathcal{C}$ is arbitrary, then the previous paragraph implies that $\mathcal{D} = \mathcal{D}_B$. Thus, we conclude that $A \in \mathcal{D}_B$ which implies that $A \cap B \in \mathcal{D}$. It now follows that $B \in \mathcal{D}_A$. This shows that $\mathcal{C} \subseteq \mathcal{D}_A$ for every $A \in \mathcal{D}$ as required. Since $\mathcal{D}_A \subseteq \mathcal{D}$ for every $A \in \mathcal{D}$ by definition, we have shown

$$\mathcal{C} \subseteq \mathcal{D}_A \subseteq \mathcal{D}$$

for every $A \in \mathcal{D}$.

The fact that $\mathcal{D}$ is the smallest class containing $\mathcal{C}$ which is closed under increasing limits and finite differences forces us, using the same argument as above, to conclude that $\mathcal{D} = \mathcal{D}_A$ for every $A \in \mathcal{D}$.

Since $\mathcal{D} = \mathcal{D}_A$ for all $A \in \mathcal{D}$, we conclude that $\mathcal{D}$ is closed under finite intersections. Furthermore, $\Omega \in \mathcal{D}$ and $\mathcal{D}$ is closed by finite differences which implies that $\mathcal{D}$ is closed under complementation. Since $\mathcal{D}$ is also closed by increasing limits, we conclude that $\mathcal{D}$ is a σ-algebra, and it is clearly the smallest σ-algebra containing $\mathcal{C}$. Thus, $\sigma(\mathcal{C}) \subseteq \mathcal{D}$ and the proof that $\mathcal{D} = \sigma(\mathcal{C})$ is complete. □

Corollary 7.2. *Let Ω be a sample space and suppose that $\mathcal{F}$ is a σ-algebra of subsets of Ω. Suppose further that $\mathbf{P}$, $\mathbf{Q}$ are two probabilities on $(\Omega, \mathcal{F})$. If $\mathbf{P}$ and $\mathbf{Q}$ agree on a class $\mathcal{C}$ which is closed under finite intersections, and $\sigma(\mathcal{C}) = \mathcal{F}$, then $\mathbf{P} = \mathbf{Q}$.*

Proof. Since $\mathcal{F}$ is a σ-algebra we know that $\Omega \in \mathcal{F}$. Since $\mathbf{P}\{\Omega\} = \mathbf{Q}\{\Omega\} = 1$ we can assume without loss of generality that $\Omega \in \mathcal{C}$. Define

$$\mathcal{D} = \{A \in \mathcal{F} : \mathbf{P}\{A\} = \mathbf{Q}\{A\}\}$$

to be the class on which $\mathbf{P}$ and $\mathbf{Q}$ agree (and note that $\emptyset \in \mathcal{D}$ and $\Omega \in \mathcal{D}$ so that $\mathcal{D}$ is non-empty). Using the definition of probability, a calculation shows that $\mathcal{D}$ is closed by finite differences and increasing limits. By assumption, we also have $\mathcal{C} \subseteq \mathcal{D}$. Therefore, since $\sigma(\mathcal{C}) = \mathcal{F}$, it follows from the monotone class theorem that $\mathcal{D} = \mathcal{F}$ as required. □