Lecture #4: There is no uniform probability on $([0,1], 2^{[0,1]})$

Our goal for today is to prove the first of the claims made last lecture, namely that there does not exist a uniform probability on the sample space $[0,1]$ with the σ-algebra $2^{[0,1]}$. Suppose that $\mathbf{P}$ is our candidate for the uniform probability on $([0,1], 2^{[0,1]})$. Motivated by our experience with elementary probability, it is desirable for such a uniform probability to satisfy $\mathbf{P}\{[a,b]\} = b-a$ for any interval $[a,b] \subseteq [0,1]$. In other words, the probability of any interval is just its length. In fact, if $0 \le a < b \le 1$, then the uniform probability should satisfy

$$\mathbf{P}\{[a,b]\} = \mathbf{P}\{(a,b)\} = \mathbf{P}\{[a,b)\} = \mathbf{P}\{(a,b]\} = b-a.$$

In particular,

$$\mathbf{P}\{a\} = 0 \quad \text{for every } 0 \le a \le 1.$$

Furthermore, the uniform probability should also satisfy countable additivity since this is one of the axioms for probability. That is, if $0 \le a_1 < b_1 < \cdots < a_n < b_n < \cdots \le 1$, then the uniform probability should also satisfy

$$\mathbf{P}\left\{\bigcup_{i=1}^{\infty}[a_i,b_i]\right\} = \sum_{i=1}^{\infty}\mathbf{P}\{[a_i,b_i]\} = \sum_{i=1}^{\infty}(b_i-a_i).$$

For instance, the probability that the outcome is in the interval $[0,1/4]$ is $1/4$, the probability the outcome is in the interval $[1/3,1/2]$ is $1/6$, and the probability that the outcome is in either the interval $[0,1/4]$ or $[1/3,1/2]$ should be $1/4+1/6=5/12$. That is,

$$\mathbf{P}\{[0,1/4]\cup[1/3,1/2]\} = \mathbf{P}\{[0,1/4]\} + \mathbf{P}\{[1/3,1/2]\} = \frac{1}{4}+\frac{1}{6} = \frac{5}{12}.$$

If $\mathbf{P}$ is to be the uniform probability on $[0,1]$, then it should also be unaffected by shifting. In particular, it should only depend on the length of the interval and not the endpoints themselves. For instance,

$$\mathbf{P}\{[0,1/4]\} = \mathbf{P}\{[1/6,5/12]\} = \mathbf{P}\{[3/4,1]\} = \frac{1}{4},$$

or, more generally,

$$\mathbf{P}\{[r,1/4+r]\} = \frac{1}{4} \quad \text{for every } 0 < r \le 3/4.$$

Note that if $3/4 < r < 1$, then $[r,1/4+r]$ is no longer a subset of $[0,1]$. But if we allow "wrapping around" then $[r,1/4+r]$ might become two disjoint intervals, each a subset of $[0,1]$, having total length $1/4$. For instance, if $r=15/16$, then $[r,1/4+r]=[15/16,19/16]$ which when "wrapped around" becomes $[0,3/16]\cup[15/16,1]$. Note that the total length of $[0,3/16]\cup[15/16,1]$ is $3/16+1/16=1/4$. That is, using finite additivity,

$$\mathbf{P}\{[0,3/16]\cup[15/16,1]\} = \frac{1}{4} = \mathbf{P}\{[0,1/4]\}.$$

We can write this (allowing for "wrapping around") using the $\oplus$ symbol so that

$$[0,1/4]\oplus r=\begin{cases}[r,1/4+r], & \text{if } 0<r\le 3/4,\\ [0,1/4+r-1]\cup[r,1], & \text{if } 3/4<r<1.\end{cases}$$

Hence, if $0<r\le 3/4$, then

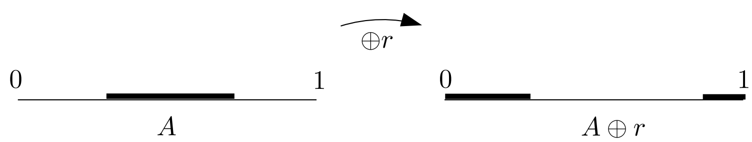
$$\mathbf{P}\left\{[0,1/4]\oplus r\right\}=\mathbf{P}\left\{[r,1/4+r]\right\}=\frac{1}{4}+r-r=\frac{1}{4}$$

while if $3/4<r<1$, then

$$\begin{aligned}\mathbf{P}\left\{[0,1/4]\oplus r\right\}=\mathbf{P}\left\{[0,1/4+r-1]\cup[r,1]\right\}&=\mathbf{P}\left\{[0,1/4+r-1]\right\}+\mathbf{P}\left\{[r,1]\right\}\\ &=\left(\frac{1}{4}+r-1\right)+(1-r)=\frac{1}{4}.\end{aligned}$$

In general, if $A\subseteq[0,1]$ is any subset of $[0,1]$, then we can define the shift of A by r for any $0<r<1$ as

$$A\oplus r=\{a+r : a\in A, a+r\le 1\}\cup\{a+r-1 : a\in A, a+r>1\}.$$



And so if $\mathbf{P}$ is to be our candidate for the uniform probability, then it is reasonable to assume that

$$\mathbf{P}\left\{A\oplus r\right\}=\mathbf{P}\left\{A\right\}$$

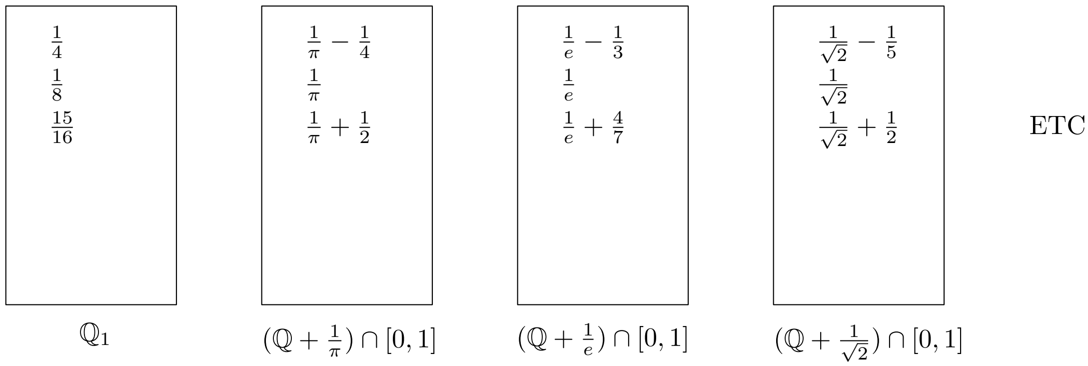
for any $0<r<1$.

To prove that no uniform probability exists for every $A\in 2^{[0,1]}$ we will derive a contradiction. Suppose that there exists such a $\mathbf{P}$. Define an *equivalence relation* on $[0,1]$ by setting $x\sim y$ iff $y-x\in\mathbb{Q}$. For instance,

$$\frac{1}{2}\sim\frac{1}{4},\quad \frac{1}{9}\sim\frac{1}{27},\quad \frac{1}{9}\sim\frac{1}{4},\quad \frac{1}{3}\not\sim\frac{1}{\pi},\quad \frac{1}{e}\not\sim\frac{1}{\pi},\quad \frac{1}{\pi}-\frac{1}{4}\sim\frac{1}{\pi}+\frac{1}{2}.$$

This equivalence relationship partitions $[0,1]$ into a disjoint union of equivalence classes (with two elements of the same class differing by a rational, but elements of different classes differing by an irrational). Let $\mathbb{Q}_1=[0,1]\cap\mathbb{Q}$, and note that there are uncountably many equivalence classes. Formally, we can write this disjoint union as

$$[0,1]=\mathbb{Q}_1\cup\left\{\bigcup_{x\in[0,1]\setminus\mathbb{Q}_1}\{(\mathbb{Q}+x)\cap[0,1]\}\right\}=\mathbb{Q}_1\cup\left\{\bigcup_{x\in[0,1]\setminus\mathbb{Q}_1}\{\mathbb{Q}_1\oplus x\}\right\}.$$



Let H be the subset of $[0,1]$ consisting of precisely one element from each equivalence class. (This step uses the Axiom of Choice.) For definiteness, assume that $0 \notin H$. Therefore, we can write $(0,1]$ as a disjoint, *countable* union of shifts of H. That is,

$$(0,1] = \bigcup_{r \in \mathbb{Q}_1, r \neq 1} \{H \oplus r\}$$

with $\{H \oplus r_i\} \cap \{H \oplus r_j\} = \emptyset$ for all $i \neq j$ which implies

$$\mathbf{P}\{(0,1]\} = \mathbf{P}\left\{ \bigcup_{r \in \mathbb{Q}_1, r \neq 1} \{H \oplus r\} \right\} = \sum_{r \in \mathbb{Q}_1, r \neq 1} \mathbf{P}\{H \oplus r\} = \sum_{r \in \mathbb{Q}_1, r \neq 1} \mathbf{P}\{H\}.$$

In other words,

$$1 = \sum_{r \in \mathbb{Q}_1, r \neq 1} \mathbf{P}\{H\}.$$

We have now arrived at our contradiction. Suppose that we wish to assign probability $p = \mathbf{P}\{H\}$ to the set H. The previous line tells us that p satisfies

$$1 = \sum_{r \in \mathbb{Q}_1, r \neq 1} p. \tag{4.2}$$

However, since p is a number between 0 and 1, there are two possibilities: (i) if $p = 0$, then

$$\sum_{r \in \mathbb{Q}_1, r \neq 1} p = \sum_{r \in \mathbb{Q}_1, r \neq 1} 0 = 0,$$

and (ii) if $0 < p \leq 1$, then

$$\sum_{r \in \mathbb{Q}_1, r \neq 1} p = \infty.$$

In either case, we see that (4.2) cannot be satisfied for *any* choice of p with $0 \leq p \leq 1$. The conclusion that we are forced to make is that we *cannot* assign a uniform probability to the set H. That is, H is *not* an event so $\mathbf{P}\{H\}$ does not exist.

We can summarize our work with the following theorem.

Theorem 4.1. *Consider the uncountable sample space* $[0,1]$ *with* σ*-algebra* $2^{[0,1]}$*, the power set of* $[0,1]$*. There does not exist a probability* $\mathbf{P} : 2^{[0,1]} \to [0,1]$ *satisfying both* $\mathbf{P}\{[a,b]\} = b - a$ *for all* $0 \leq a \leq b \leq 1$*, and* $\mathbf{P}\{A \oplus r\} = \mathbf{P}\{A\}$ *for all* $A \subseteq [0,1]$ *and* $0 < r < 1$*.*

In other words, it is not possible to define a uniform probability $\mathbf{P}\{A\}$ for *every* set $A \subseteq [0,1]$. The fact that there exists a set $H \subseteq [0,1]$ such that $\mathbf{P}\{H\}$ does not exist means that the σ-algebra $2^{[0,1]}$ is simply too big! Instead, as we shall see, the "correct" σ-algebra to use is $\mathcal{B}_1$, the Borel σ-algebra of $[0,1]$. Thus, our next goal, which will still take several lectures to accomplish, is to construct the uniform probability on $([0,1], \mathcal{B}_1)$.