Statistics 451 (Fall 2013)
Basic Set Theory

Let $\mathbb{N}$ denote the set of natural numbers, namely $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Let $\mathbb{Z}$ denote the set of integers, namely $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The non-negative integers are $\{0, 1, 2, 3, \ldots\} = \{0\} \cup \mathbb{N}$.

Let $\mathbb{Q}$ denote the set of rational numbers, namely those numbers that can be written as the ratio of an integer to a natural number, say

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0 \right\}.$$

Let $\mathbb{R}$ denote the set of real numbers. There are a number of ways to construct $\mathbb{R}$. One is as the completion of the rational numbers.

A number is called irrational if it is not rational, namely x is irrational iff $x \in \mathbb{R} \setminus \mathbb{Q}$.

A set is called countable if it can be put into a one-to-one correspondence with $\mathbb{N}$. In other words, a set E is countable iff there exists a bijective function $f : E \to \mathbb{N}$. Recall that a bijective function, or bijection, is both one-to-one (also called injective) and onto (also called surjective).

Note that $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

A set is called uncountable if it is not countable. The set $\mathbb{R}$ is uncountable as is the unit interval $[0, 1]$. This can be shown using Cantor's diagonal argument.

A set that contains a finite number of elements, say $E = \{x_1, \ldots, x_n\}$ with $n < \infty$, is called finite or discrete. We define the cardinality of such a set as $|E| = n$.

Note that a countable set E contains infinitely many elements. Thus, we say that a set is at most countable if it is either countable or discrete. The cardinality of a countable set is defined to be $\aleph_0$ (the Hebrew letter aleph, subscript 0, read aleph-nought); that is, $|\mathbb{N}| = \aleph_0$.

Note that an uncountable set also contains infinitely many elements. The cardinality of the real numbers is $|\mathbb{R}| = \mathbf{c}$ (for continuum). Cantor showed that $\aleph_0 < \mathbf{c}$.

The cardinality of the power set of the natural numbers is $|2^{\mathbb{N}}| = 2^{\aleph_0}$. Cantor's diagonal argument can also be used to show that $\mathbf{c} = 2^{\aleph_0}$.

The continuum hypothesis says that there is no set S for which $\aleph_0 < |S| < \mathbf{c}$. Paul Cohen and Kurt Gödel proved that the continuum hypothesis is independent of the Zermelo-Fraenkel axioms of set theory and the axiom of choice (ZFC).